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# REVERSING THE AHLFORS' ESTIMATE

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ABSTRACT. It is shown that the Ahlfors' inequality for a complete Hermitian metric on the punctured disc with Gaussian curvature bounded below zero can be partially reversed for the averaged metric which is also complete and has the same curvature properties.

The Ahlfors' estimate is well known:

**Proposition.** *Let $dS^2 = h|dw|^2$ be a complete Hermitian metric on the punctured disc. If the Gaussian curvature is bounded above by a negative constant $b$, then $h \leq C/|W|^2(\ln |W|^{-1})^2$ for a constant dependent only on $b$.*

It is the basis for much of hyperbolic complex analysis [2] and has many applications in transcendental algebraic geometry [1]. In the latter context I was led to the question of reversing the Ahlfors' estimate (cf. [3, Appendix II]).

Let $\oint f = (1/2\pi) \int_0^{2\pi} f(r, \theta)\, d\theta$ for a function on the punctured disc.

**Proposition.** *Let $dS^2 = h|dw|^2$ be a Hermitian metric on the punctured disc $\Delta^* = \{W \in \mathbf{C} | 0 < |W| < 1\}$ with Gaussian curvature bounded above by a negative constant $-b$. $(\oint h^p)^{1/p} |dw|^2$ with $1 \leq p < \infty$ satisfies the same curvature condition and is complete if $h|dw|^2$ is. Further if $dS^2$ is complete, then given any proper subdisc $\Delta^*(r) = \{W \in \mathbf{C} | 0 < |W| < r < 1\}$ and any $\epsilon > 0$, there exists a constant $C > 0$ such that $(\oint h^p)^{1/p} \geq C/r^{2-\epsilon}$.*

**Proof.** Let $\Delta\phi = \partial_x^2\phi + \partial_y^2\phi$ and let $|\nabla\phi|^2 = (\partial_x\phi)^2 + (\partial_y\phi)^2 = (\partial_r\phi)^2 + (1/r)^2(\partial_\theta\phi)^2$. The curvature condition $\Delta \ln h \geq bh$ can thus be written $\Delta h \geq |\nabla h|^2/h + bh^2$. Note that $\Delta\oint = \oint\Delta$.

Therefore

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$$\Delta \ln \left(\oint h^p\right)^{1/p} = \frac{1}{p}\left[\frac{\Delta \oint h^p}{\oint h^p} - \frac{|\nabla \oint h^p|^2}{(\oint h^p)^2}\right]$$

$$= \frac{1}{p}\left[\frac{p \oint h^{p-1}\Delta h}{\oint h^p} + p(p-1)\frac{\oint h^{p-2}|\nabla h|^2}{\oint h^p} - p^2\frac{(\oint [\partial_r h] h^{p-1})^2}{(\oint h^p)^2}\right]$$

$$\geq b\frac{\oint h^{p+1}}{\oint h^p} + p\left[\frac{\oint |\nabla h|^2 h^{p-2}}{\oint h^p} - \left(\frac{\oint (\partial_r h) h^{p-1}}{\oint h^p}\right)^2\right]$$

$$\geq b\left(\oint h^p\right)^{1/p} + p\left[\frac{(\oint h^p)(\oint |\nabla h|^2 h^{p-2}) - (\oint [(\partial_r h) h^{(p-2)/2}] h^{p/2})^2}{(\oint h^p)^2}\right]$$

$$\geq b\left(\oint h^p\right)^{1/p}.$$

The last two inequalities are straightforward consequences of Hölder's inequality.

Since the $L^1$ norm with respect to $d\theta/2\pi$ is less than or equal to the $L^p$ norm for $p \geq 1$ it will suffice to prove completeness and the inequality for $p = 1$.

$\oint h$ depends only on $r$. Therefore any path between $\rho_0 e^{i\theta_0}$ and $\rho_1 e^{i\theta_1}$ is greater in length than the radial line segment between $\rho_0$ and $\rho_1$. From this it is clear that completeness will follow from showing that $\int_0^{r_0} (\oint h)^{1/2}\, dr = \infty$ and $\int_{r_0}^1 (\oint h)^{1/2}\, dr = \infty$ for $0 < r_0 < 1$.

Note that $(\oint h)^{1/2} \geq \oint h^{1/2}$. Therefore

$$\int_0^{r_0} \left(\oint h\right)^{1/2} dr \geq \int_0^{r_0} \left(\oint h^{1/2}\right) dr = \oint \left(\int_0^{r_0} h^{1/2}\, dr\right) = \infty$$

if $h|dw|^2$ is complete. The case of $\int_{r_0}^1 (\oint h)^{1/2}\, dr$ is the same.

In what follows it may be assumed without loss of generality that $h$ is already averaged and is thus a function of $r$ alone.

$\ln h(r) \geq C \ln(1/r)$ or $< C \ln(1/r)$ for $0 < C < 2$ and each $r$. Assume that there was a sequence $r_n \to 0$ with $r_n > r_{n+1} > \cdots$ such that $\ln h(r_n) < C \ln(1/r_n)$. Since $h(r)$ and $\ln(1/r)$ are functions of $r$, these estimates hold on concentric circles. Since $\ln h(r)$ is subharmonic and $\ln(1/r)$ is harmonic this holds on the annuli between any two concentric circles. Therefore

$\ln h(r) < C \ln(1/r)$ for $r < r_1$. This contradicts completeness:

$$\int_0^{r_1} h^{1/2} \leq \int_0^{r_1} dr/r^{c/2} = (2/(2-c))\,[r^{-(c/2-1)}]_0^{r_1} < \infty$$

since $c < 2$. Therefore given any $\epsilon > 0$ and any proper subdisc, there exists a $C' > 0$ such that $h(r) \geq C'/r^{2-\epsilon}$. Q.E.D.

Except for differentiability the case $p = \infty$ also follows.

*Question One.* Can the estimate be improved to $\oint h \geq C/r^2(\ln(1/r))^{2+\epsilon}$, $\epsilon > 0$, on some proper subdisc?

This would be equivalent to showing that $h(y) \geq C/y^{2+\epsilon}$ for $y > y_0 > 0$ where $h(y)|dz|^2$ is a complete metric on the upper half plane with Gaussian curvature bounded above by a negative constant and $h$ dependent only on $y$. This has resisted many attempts and I half suspect it cannot be improved.

*Question Two.* How bad can things be if the metrics are unaveraged?

*Question Three.* What can be said about $p < 1$ or even $e^{\oint \ln^+ h}|dx|^2$?

The completeness is hard to show. This is especially interesting when one notes that the above estimates can be arranged

$$C'/r^{2-\epsilon} \leq \|h\|_1(r) \leq \|h\|_p(r) \leq \|h\|_\infty(r) \leq C/r^2(\ln(1/r))^2.$$

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