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ON COMPACT* SPACES AND COMPACTIFICATIONS

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ABSTRACT. The space βX of Z-ultrafilters on X with the standard filter space topology is shown to be compact*. Without considering the reflection associated with compact* spaces, we also prove that products of compact* spaces are compact*, in response to a request for a direct proof.

Introduction. Compact* spaces were defined by W. W. Comfort [2] as completely regular Hausdorff spaces X for which every maximal ideal in $C^*(X)$ is fixed. He proved without the axiom of choice that every completely regular Hausdorff space X can be densely C^*-embedded in a compact* space βX and deduced that products of compact* spaces are compact*. The problem of proving directly the productivity of compactness* was raised and left open.

In §1 of this note we establish a one-to-one correspondence between the maximal ideals of $C^*(X)$ and the Z-ultrafilters on X without the axiom of choice and show that X is compact* if and only if every Z-ultrafilter on X converges. We then have a topological method for the study of compactness*.

We use the above method to show in §2 that the space βX of Z-ultrafilters on X [3] is compact* and that the classical characterizations of βX [3] hold independently of the axiom of choice.

Finally, in §3 we prove directly that products of compact* spaces are compact* and that closed subspaces of compact* spaces are compact*. The method of proof differs from that of §2 in that it involves a consideration of maximal ideals in rings of real valued bounded continuous functions. W. W. Comfort's theorem referred to above is a consequence of the results of this section.

An alternative construction of βX has recently been given by R. E. Chandler [1].

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Our standard reference is L. Gillman and M. Jerison's *Rings of continuous functions* [3]. □ indicates the end of a proof.

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1. **Alternative characterization of compactness*.** We establish directly a one-to-one correspondence between the maximal ideals of $C^*(X)$ (and $C(X)$) and the Z-ultrafilters on X without the axiom of choice. In what follows, let C denote $C^*(X)$ or $C(X)$.

Proposition 1. *Let M be a maximal ideal in C. Let $A(M)$ consist of the nonempty zero sets Z such that* $\inf |m|[Z] = 0$ *for all m in M. Then $A(M)$ is a Z-ultrafilter on X.*

Proof. It is clear that zero sets which contain a zero set in $A = A(M)$ are also in A. Also, by definition of A, if $Z \in A$, then Z intersects every $Z_\delta(m) = |m|^{\leftarrow}[0, \delta]$, $m \in M$, $\delta > 0$. Moreover, if $m \in M$, then $Z_\delta(m) \in A$ for every $\delta > 0$. Otherwise there is $m' \in M$ and $\delta' > 0$ such that $Z_\delta(m)$ and $Z_{\delta'}(m')$ are disjoint, but then $m^2 + (m')^2 \geq \min(\delta^2, (\delta')^2)$ which is impossible since M contains no invertible elements. We now show that A is a filter. Suppose Z_0, Z_1 are in A, and that $Z_0 \cap Z_1$ is not in A. Then there is $m \in M$ and $\delta > 0$ such that $Z_0 \cap Z_1 \cap Z_\delta(m) = \emptyset$. Thus Z_0 and $Z_1 \cap Z_\delta(m)$ are disjoint zero sets, so there is $h: X \to [0, 2]$ such that $h = 0$ on Z_0 and $h = 2$ on $Z_1 \cap Z_\delta(m)$. Now $h \in M$, otherwise $2 = kh + m''$, for some $k \in C$ and $m'' \in M$. Hence $m'' = 2$ on Z_0, which is impossible since then $Z_1(m'')$ and Z_0 would be disjoint, contradicting $Z_0 \in A$. But now we get that $Z_1(h)$ is disjoint from $Z_1 \cap Z_\delta(m)$, which is not possible as $Z_1(h) \cap Z_1 \cap Z_\delta(m) \supset Z_1 \cap Z_k(h^2 + m^2)$, where $k = \min\{\delta^2, 1\}$, since $Z_1 \in A$ and $h^2 + m^2 \in M$. Thus $Z_0 \cap Z_1 \in A$ if $Z_0, Z_1 \in A$. Finally the Z-ultrafilter property is an immediate consequence of the fact that $Z_\delta(m) \in A$ for all $m \in M$, and $\delta > 0$. □

The inverse correspondence has a more straightforward proof which we omit.

Proposition 2. *Let A be a Z-ultrafilter on X. Let $M(A)$ consist of the functions m in C such that* $\inf |gm|[Z] = 0$ *for all $g \in C$ and $Z \in A$. Then $M(A)$ is a maximal ideal in C.*

It is interesting to note that it is the maximality of A that ensures that

$M(A)$ is closed under addition. Also, if $C = C^*(X)$, then $M(A)$ consists of those m such that $\inf |m|[Z] = 0$, for all $Z \in A$.

The requirements of maximality in both propositions cannot be dropped as shown in the following example.

Example. Let $X = \{0, \pm 1, \pm 2, \cdots\}$ have the discrete topology.

(a) Let M be the ideal generated by $j(n) = 1/n$. Then $\{-1, -2, \cdots\}$ and $\{1, 2, \cdots\}$ are disjoint zero sets in $A(M)$.

(b) Let F be the filter generated by the sets $A_n = \{x \in X \mid x^2 \geq n^2\}$. Then $M(F)$ contains both functions f, g given below but does not contain $f + g$, where $f(n) = 1/n$ if $n > 0$ and $f(n) = 1$ if $n \leq 0$; $g(n) = 1$ if $n \geq 0$ and $g(n) = 1/n$ if $n < 0$.

Proposition 3. *There is a one-to-one correspondence between the maximal ideals of* C *and the* Z*-ultrafilters on* X *given by* $M(A(M)) = M$ *and* $A(M(A)) = A$.

If $C = C(X)$, the above correspondence coincides with the Z-correspondence in [3].

It is now simple to prove an alternative characterization of compactness*. Again we omit the proof.

Proposition 4. X *is compact* if and only if every* Z*-ultrafilter on* X *converges.*

2. **Characterization of** βX. The following proposition is analogous to Theorem 6.4 of [3] and serves as a preparation for the characterizations of βX given in Theorem 1.

Proposition 5. *Let* T *be a topological space and* X *a subspace such that every point of* T *is the limit of a* Z*-ultrafilter on* X. *The statements* (1) *to* (4) *are equivalent and* (4) *implies* (5).

(1) *Every continuous map into a compact* space* Y *has an extension to a continuous map from* T *into* Y.

(2) X *is* C^**-embedded in* T.

(3) *Any two disjoint zero sets in* X *have disjoint closures in* T.

(4) *For any two zero sets* Z_1, Z_2 *in* X, $\mathrm{cl}_T(Z_1 \cap Z_2) = \mathrm{cl}_T Z_1 \cap \mathrm{cl}_T Z_2$.

(5) *Every point* p *of* T *is the limit of a unique* Z*-ultrafilter* A_p *in* X.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) without the axiom of choice.

(3) $\Rightarrow$ (4): It follows from (3) that if A is a Z-ultrafilter on X which converges to p and if $p \in \mathrm{cl}_T Z$, then $Z \in A$. Thus, if $p \in \mathrm{cl}_T Z_1 \cap \mathrm{cl}_T Z_2$ then $Z_1, Z_2 \in A$, hence $Z_1 \cap Z_2 \in A$, so that $p \in \mathrm{cl}_T(Z_1 \cap Z_2)$. Thus (4) is proved.

It is clear that (4) $\Rightarrow$ (5). We now prove that (3) $\Rightarrow$ (1). Let $f\colon X \to Y$ be continuous and suppose Y is compact*. Let $p \in T$ and let A_p be the unique Z-ultrafilter on X which converges to p. Let ${}_pA$ be the family of zero sets $E \subset Y$ which intersect every zero set F such that $f^{\leftarrow}[F] \in A_p$. We show that ${}_pA$ is a Z-ultrafilter.

Suppose $E_0, E_1 \in {}_pA$. If $E_0 \cap E_1 \notin {}_pA$, then there is F such that $f^{\leftarrow}[F] \in A_p$ and $E_0 \cap E_1 \cap F = \emptyset$. Let $h\colon Y \to [0, 2]$ be such that $h = 0$ on E_0 and $h = 2$ on $E_1 \cap F$. Now $h^{\leftarrow}[1, 2] \cap E_0 = \emptyset$, so $f^{\leftarrow}[h^{\leftarrow}[1, 2]] \notin A_p$, hence $f^{\leftarrow}[h^{\leftarrow}[0, 1]] \in A_p$. But then E_1 is disjoint from the zero set $F_1 = h^{\leftarrow}[0, 1] \cap F$ and $f^{\leftarrow}[F_1] \in A_p$, which is impossible. Hence ${}_pA$ is closed under finite intersections. It is simple to prove that ${}_pA$ is in fact a Z-ultrafilter. Now Y is compact* so ${}_pA$ converges to a unique point, $\bar{f}(p)$, say. Thus $\bar{f}(p)$ is the only element of $\bigcap\{F \mid F \in {}_pA\}$. If $x \in X$, then $x \in \bigcap\{E \mid E \in A_x\}$ and $f(x) \in \bigcap\{F \mid F \in {}_xA\}$, otherwise there is $F \in {}_xA$ such that $f(x) \notin F$ so that there is a zero set H which contains $f(x)$ and is disjoint from F, but this is not possible since F must intersect H by definition of ${}_xA$. Thus $\bar{f}(x) = f(x)$, if $x \in X$. Finally, the continuity of $\bar{f}$. Note that if F is a zero set in Y which is not in ${}_pA$, then by definition of ${}_pA$, there is a zero set E disjoint from F such that $f^{\leftarrow}[E] \in A_p$ and (3) implies $\mathrm{cl}_T f^{\leftarrow}[E] \cap \mathrm{cl}_T f^{\leftarrow}[F] = \emptyset$, and we have remarked that $p \in \mathrm{cl}_T f^{\leftarrow}[E]$, hence $p \notin \mathrm{cl}_T f^{\leftarrow}[F]$. The proof that $\bar{f}$ is continuous can now be completed as in [3]. □

Note. The above proposition requires a more elaborate proof than that of Theorem 6.4 of [3]. This is due to two factors. Firstly we do not assume T is completely regular, so the proof of (3) $\Rightarrow$ (4) in [3] does not apply. Secondly, we have not been able to prove that (5) $\Rightarrow$ (1) without the axiom of choice. However (5) does imply (1) under an added assumption on how X is embedded in T, as shown in Proposition 6.

Proposition 6. *Let X be dense in T and such that if Z is a zero set in X and $p \in \mathrm{cl}_T Z$, then there is a Z-ultrafilter on X which contains Z and converges to p. Then, any two disjoint zero sets in X have disjoint closures in T if and only if every point of T is the limit of a unique Z-ultrafilter on X.*

Proof. The hypotheses of the theorem ensure that every point of T is the limit of a Z-ultrafilter on X, so one implication has been proved in Proposition 5. Conversely, suppose $p \in \mathrm{cl}_T Z_1 \cap \mathrm{cl}_T Z_2$. Let A_p be the unique Z-ultrafilter on X converging to p. By hypothesis, Z_1 and Z_2 are both members of A_p, hence Z_1 and Z_2 are not disjoint. □

Proposition 7. *Let T be a topological space and X a subspace such that every point of T is the limit of some Z-filter on X. If the sets $\mathrm{cl}_T Z$, Z a zero set in X, form a base for the closed sets of T and any of* (1) *to* (4) *hold, then T is completely regular.*

Proof. Let $p \in T$ and $F \subset T$ be a closed set not containing p. By hypothesis there is a zero set Z in X such that $F \subset \mathrm{cl}_T Z$ and $p \notin \mathrm{cl}_T Z$. Hence there is $Z_1 \in A_p$ such that $Z_1 \cap \mathrm{cl}_T Z = \emptyset$. Thus $Z_1 \cap Z = \emptyset$, so there is a continuous map $h: X \to [0, 1]$ such that $h = 0$ on Z and $h = 1$ on Z_1. But X is C^*-embedded in T so h has an extension $\bar{h}$ to T. Then $\bar{h} = 1$ on $\mathrm{cl}_T Z_1$ and $\bar{h} = 0$ on $\mathrm{cl}_T Z$, proving complete regularity of T, since $p \in \mathrm{cl}_T Z_1$. □

Proposition 8. *If X is C^*-embedded as a dense subspace of a completely regular space T and every Z-ultrafilter on X converges in T, then every Z-ultrafilter on T converges.*

Proof. Let A be a Z-ultrafilter in T. Let A_0 denote the family of zero sets in X which intersect every $Z_\delta(f) = |f|^{\leftarrow}[0, \delta]$, where $Z(f) \in A$. The proof that A_0 is a Z-ultrafilter on X, is analogous to the proof that ${}_pA$ is a Z-ultrafilter in Proposition 5, (3) $\Rightarrow$ (1). Let $p \in T$ be the limit of A_0. It is easy to see that A also converges to p. □

We can now prove that there is a $*$-compactification βX of X, as in [3].

Theorem 1. *For every completely regular Hausdorff space X there is a compact* space βX containing X as a dense subspace with the following equivalent properties.*

(1) *Every continuous map into a compact* space Y has an extension to a continuous map from βX into Y.*

(2) *X is C^*-embedded in βX.*

(3) *Any two disjoint zero sets in X have disjoint closures in βX.*

(4) *For any two zero sets Z_1, Z_2 in X, $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) = \mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2$.*

(5) *Every point p of βX is the limit of a unique Z-untrafilter in X, A_p.*

Proof. Let βX be the set of all Z-ultrafilters on X. For each zero set Z in X, define $p \in \bar{Z}$ if $Z \in p$, where $p \in \beta X$. As shown in [3], the sets $\bar{Z}$ form a base for closed sets, $\mathrm{cl}_{\beta X} Z = \bar{Z}$ and $\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \mathrm{cl}_{\beta X}(Z_2 \cap Z_2)$. By Propositions 5 and 6 it follows that βX has all the properties (1) to (5) stated in the theorem and these properties are equivalent. The proof that βX is Hausdorff is the same as in [3]. By Propositions 7 and 8 it follows that βX is compact*. □

It is interesting to note that the characterizations of the maximal ideals of $C^*(X)$ and $C(X)$ using the points of βX (see [3]) also hold without the axiom of choice.

3. **Products and closed subspaces of compact* spaces.** In [2] W. W. Comfort posed the problem of giving a direct proof that products of compact* spaces are compact*. We could give such a direct proof. In fact our proof shows that products of compact* spaces are compact* and that products of realcompact spaces are realcompact all at once. As before, let C denote $C^*(X)$ or $C(X)$.

Definition. A completely regular space is C-compact if every maximal ideal M such that C/M is isomorphic to $\mathbf{R}$ is fixed.

Note that when $C = C(X)$, C-compact is identical to realcompact. When $C = C^*(X)$, C-compact and compact* are identical, since $C^*(X)/M \cong \mathbf{R}$ as shown by W. W. Comfort in [2].

Theorem 2. *Products of C-compact spaces are C-compact.*

Proof. Suppose X_i is C-compact for each i in a set I. Let $X = \Pi X_i$. Suppose X is not empty and let M be a maximal ideal in C ($= C^*(X)$ or $C(X)$). Let $\pi_i\colon X \to X_i$ denote the projection map and $\pi_i^*\colon C_i \to C$, the induced ring homomorphism ($C_i = C^*(X_i)$ or $C(X_i)$). Let q denote the quotient map $q\colon C \to C/M \cong \mathbf{R}$. Then $q_i = q \circ \pi_i^*$ is a ring homomorphism from C_i *onto* $\mathbf{R}$ for each i (note that $q_i(c) = q(c) = c$ for all c in $\mathbf{R}$). Hence $M_i = q_i^{\leftarrow}[0] = \pi_i^{*\leftarrow}[M]$ is a maximal ideal in C_i. Since each X_i is C-compact, it follows that there is an element $x_0 \in X$ such that $M_i = \{f \in C_i | f(\pi_i(x_0)) = 0\}$. We show that $M = \{f \in C | f(x_0) = 0\}$, or equivalently, that $q(f) = 0$ iff $f(x_0) = 0$. First observe that if $f(x_0) \neq 0$, then $f^2 + m$ is invertible in C for some $m \in M$. For suppose $f^2(x_0) = 2\delta > 0$, then there are open sets V_{ij} in X_{ij}, $j = 1, 2, \cdots, n$, such that $x_0 \in V = \bigcap \pi_{ij}^{\leftarrow}[V_{ij}]$ and $f^2 > \delta$ on V.

Let $g_{ij} \geqq 0$ be such that $g_{ij}(x_{ij}) = 0$ and $g_{ij} = 1$ off V_{ij}, where $x_{ij} = \pi_{ij}(x_0)$ fixes M_{ij}. Then $g_{ij} \in M_{ij}$, hence $\pi_{ij}^*(g_{ij}) = g_{ij} \circ \pi_{ij} \in M$. Now the function $g = f^2 + \Sigma g_{ij} \circ \pi_{ij}$ is bounded away from zero, hence invertible in C. As a consequence, we have that $q(f) = 0$ implies $f(x_0) = 0$.

For the converse implication, suppose $f(x_0) = 0$ and $q(f) \neq 0$. Then $f \notin M$, so that $1 = kf + m$ for some $k \in C$ and some $m \in M$. Then $m(x_0) = 1$. By above there is $m' \in M$ such that $m^2 + m'$ is invertible in C, which is impossible. □

Proposition 9. *Closed subspaces of C-compact spaces are C-compact.*

Proof. Let A be a closed subspace of X and $i: A \to X$ the injection map. There is an induced ring homomorphism $j: C_X \to C_A$ ($C_X = C(X)$ or $C^*(X)$, $C_A = C(A)$ or $C^*(A)$, respectively) given by $j(f) = f \circ i$. Let M be a maximal ideal in C_A, and q the quotient map $q: C_A \to C_A/M \cong \mathbf{R}$. Let $p = q \circ j$, then p is a ring homomorphism *onto* $\mathbf{R}$ since $p(c) = c$ for all $c \in \mathbf{R}$. Hence $M_1 = \ker p = j^{\leftarrow}[M]$ is a maximal ideal in C_X. X is C-compact so there is $x \in X$ such that $M_1 = \{f \in C_X | f(x) = 0\}$. Then $x \in A$, otherwise there is $h: X \to [0, 1]$, continuous, such that $h(x) = 0$, $h = 1$ on A. But then $h \in M_1$, so that $j(h) = h \circ i \in M$, which is impossible since $h \circ i = 1$. □

That the category of C-compact spaces is reflective now follows from the general theory of reflections because this category is closed under taking products and closed subspaces.

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