PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

A GENERALIZATION OF BANACH'S CONTRACTION PRINCIPLE

Lj. B. ĆIRIĆ

ABSTRACT. Let $T: M \to M$ be a mapping of a metric space (M, d) into itself. A mapping T will be called a quasi-contraction iff $d(Tx, Ty) \le q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$ for some $q < 1$ and all $x, y \in M$. In the present paper the mappings of this kind are investigated. The results presented here show that the condition of quasi-contractivity implies all conclusions of Banach's contraction principle. Multi-valued quasi-contractions are also discussed.

1. **Introduction.** The well-known Banach's contraction mapping principle states that if $T: M \to M$ is a contraction on M (i.e. $d(Tx, Ty) \leq q \cdot d(x, y)$ for some $q < 1$ and all $x, y \in M$) and M is complete, then

(1°) T has a unique fixed point u in M,

(2°) $\lim_n T^n x = u$, and

(3°) $d(T^n x, u) \leq q^n (1-q)^{-1} d(x, Tx)$ for every $x \in M$.

A number of generalizations of this result have appeared [1], [2], [3], [7], [8], [9], [12]. In [2] we considered generalized contractions, defined as follows.

A mapping $T: M \to M$ is said to be a *generalized contraction* iff for every $x, y \in M$ there exist nonnegative numbers q, r, s and t, which may depend on both x and y, such that $\sup\{q + r + s + 2t: x, y \in M\} < 1$ and

$$\text{(A)} \qquad \begin{aligned} d(Tx, Ty) \leq q \cdot d(x, y) &+ r \cdot d(x, Tx) \\ &+ s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)]. \end{aligned}$$

S. Nadler [10] has extended Banach's contraction principle to multi-valued contractions. Many extensions of Nadler's result have been derived in recent years [4], [6], [11], [13]. In [4] we proved some fixed-point theorems for a class of multi-valued generalized contractions—the maps which include the single-valued generalized contractions.

Received by the editors January 17, 1973.

AMS (MOS) subject classifications (1970). Primary 54E40, 54H25, 47H10; Secondary 54C30.

Key words and phrases. Quasi-contractions, multi-valued quasi-contractions, fixed-point theorems.

The purpose of this paper is to extend some results concerning generalized contractions of [2] and [4] to quasi-contractions. In §2 fixed-point theorems for single-valued quasi-contractions are proved and an example is given to show that the results established here are indeed extensions. In §3 it is shown that for multi-valued quasi-contractions a similar result is valid.

2. **Quasi-contractions.** Let T be a mapping of a metric space M into itself. For $A \subset M$ let $\delta(A) = \sup\{d(a, b): a, b \in A\}$ and for each $x \in M$, let

$$O(x, n) = \{x, Tx, \cdots, T^n x\}, \qquad n = 1, 2, \cdots,$$
$$O(x, \infty) = \{x, Tx, \cdots\}.$$

A space M is said to be *T-orbitally complete* iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in M$ converges in M (cf. [5]).

Definition 1. *A mapping $T: M \longrightarrow M$ of a metric space M into itself is said to be a quasi-contraction iff there exists a number q, $0 \leq q < 1$, such that*

(B) $d(Tx, Ty) \leq q \cdot \max\{d(x, y);\ d(x, Tx);\ d(y, Ty);\ d(x, Ty);\ d(y, Tx)\}$

holds for every $x, y \in M$.

It is clear that condition (A) implies (B). The following example shows that a quasi-contraction need not be a generalized contraction.

Example. Let

$$M_1 = \{m/n: m = 0, 1, 3, 9, \cdots;\ n = 1, 4, \cdots, 3k+1, \cdots\},$$

$$M_2 = \{m/n: m = 1, 3, 9, 27, \cdots;\ n = 2, 5, \cdots, 3k+2, \cdots\},$$

and let $M = M_1 \cup M_2$ with the usual metric. Define $T: M \longrightarrow M$ by

$$Tx = 3x/5, \quad \text{for } x \in M_1,$$
$$= x/8, \quad \text{for } x \in M_2.$$

The mapping T is a quasi-contraction with $q = 3/5$. Indeed, if both x and y are in M_1 or in M_2, then $d(Tx, Ty) \leq (3/5)d(x, y)$. Now let x be, for example, in M_1 and y in M_2. Then

$$x > \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5}\left(x - \frac{5}{24}y\right) \leq \frac{3}{5}\left(x - \frac{1}{8}y\right) = \frac{3}{5}d(x, Ty);$$

$$x < \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5}\left(\frac{5}{24}y - x\right) \leq \frac{3}{5}(y - x) = \frac{3}{5}d(x, y).$$

Therefore, T on M satisfies the condition

$$d(Tx, Ty) \leq (3/5)\max\{d(x, y);\ d(x, Ty);\ d(y, Tx)\}$$

and hence (B).

To show that T is not a generalized contraction on M, let $x = 1$ and $y = \frac{1}{2}$. Then we have

$$q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t[d(x, Ty) + d(y, Tx)]$$

$$= q \cdot \frac{1}{2} + r \cdot \frac{2}{5} + s \cdot \frac{7}{16} + t \cdot \frac{83}{80}$$

$$< (q + r + s + 2t) \cdot \frac{83}{160} < \frac{83}{160} < \frac{43}{80} = d(Tx, Ty),$$

as $q + r + s + 2t < 1$, and we see that condition (A) is not satisfied.

Before stating the fixed-point theorem for quasi-contractions we shall prove two lemmas on these mappings. The first of these lemmas is fundamental.

Lemma 1. *Let T be a quasi-contraction on M and let n be any positive integer. Then for each $x \in M$ and all positive integers i and j, $i, j \in \{1, 2, \cdots, n\}$ implies $d(T^i x, T^j x) \leq q \cdot \delta[O(x, n)]$.*

Proof. Let $x \in M$ be arbitrary, let n be any positive integer and let i and j satisfy the condition of Lemma 1. Then $T^{i-1}x, T^i x, T^{j-1}x, T^j x \in O(x, n)$ (where it is understood that $T^0 x = x$) and since T is a quasi-contraction, we have

$$\begin{aligned} d(T^i x, T^j x) &= d(TT^{i-1}x, TT^{j-1}x) \\ &\leq q \cdot \max\{d(T^{i-1}x, T^{j-1}x);\ d(T^{i-1}x, T^i x);\ d(T^{j-1}x, T^j x); \\ &\qquad d(T^{i-1}x, T^j x);\ d(T^i x, T^{j-1}x)\} \\ &\leq q \cdot \delta[O(x, n)], \end{aligned}$$

which proves the lemma.

Remark. From this lemma it follows that if T is a quasi-contraction and $x \in M$, then for every positive integer n there exists a positive integer $k \leq n$, such that $d(x, T^k x) = \delta[O(x, n)]$.

Lemma 2. *If T is a quasi-contraction on M, then*

$$\delta[O(x, \infty)] \leq (1/(1 - q))d(x, Tx)$$

holds for all $x \in M$.

Proof. Let $x \in M$ be arbitrary. Since $\delta[O(x, 1)] \leq \delta[O(x, 2)] \leq \cdots$, we see that $\delta[O(x, \infty)] = \sup\{\delta[O(x, n)]: n \in N\}$. The lemma will follow if we show that $\delta[O(x, n)] \leq (1/(1-q))d(x, Tx)$ for all $n \in N$.

Let n be any positive integer. From the remark to the previous lemma, there exists $T^k x \in O(x, n)$ $(1 \leq k \leq n)$ such that $d(x, T^k x) = \delta[O(x, n)]$. Applying a triangle inequality and Lemma 1, we get

$$d(x, T^k x) \leq d(x, Tx) + d(Tx, T^k x) \leq d(x, Tx) + q \cdot \delta[O(x, n)]$$
$$= d(x, Tx) + q \cdot d(x, T^k x).$$

Therefore, $\delta[O(x, n)] = d(x, T^k x) \leq (1/(1-q))d(x, Tx)$. Since n was arbitrary, the proof is completed.

Now we can state our main result.

Theorem 1. *Let T be a quasi-contraction on a metric space M and let M be T-orbitally complete. Then*

(a) *T has a unique fixed point* u *in M,*

(b) $\lim_n T^n x = u$, *and*

(c) $d(T^n x, u) \leq (q^n/(1-q))d(x, Tx)$ *for every* $x \in M$.

Proof. Let x be an arbitrary point of M. We shall show that the sequence of iterates $\{T^n x\}$ is a Cauchy sequence.

Let n and m $(n < m)$ be any positive integers. Since T is a quasi-contraction, it follows from Lemma 1 that

$$d(T^n x, T^m x) = d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \leq q \cdot \delta[O(T^{n-1}x, m-n+1)].$$

According to the remark to Lemma 1, there exists an integer k_1, $1 \leq k_1 \leq m-n+1$, such that

$$\delta[O(T^{n-1}x, m-n+1)] = d(T^{n-1}x, T^{k_1}T^{n-1}x).$$

Again, by Lemma 1, we have

$$d(T^{n-1}x, T^{k_1}T^{n-1}x) = d(TT^{n-2}x, T^{k_1+1}T^{n-2}x)$$
$$\leq q \cdot \delta[O(T^{n-2}x, k_1 + 1)]$$
$$\leq q \cdot \delta[O(T^{n-2}x, m-n+2)].$$

Therefore, we have the following system of inequalities.

$$d(T^n x, T^m x) \leq q \cdot \delta[O(T^{n-1}x, m-n+1)] \leq q^2 \cdot \delta[O(T^{n-2}x, m-n+2)].$$

Proceeding in this manner, we obtain

$$d(T^n x,\ T^m x) \leq q \cdot \delta[O(T^{n-1}x,\ m-n+1)] \leq \cdots \leq q^n \cdot \delta[O(x,\ m)].$$

Then it follows from Lemma 2 that

$$d(T^n x,\ T^m x) \leq (q^n/(1-q))d(x,\ Tx). \tag{1}$$

Since $\lim_n q^n = 0$, $\{T^n x\}$ is a Cauchy sequence.

Again, M being T-orbitally complete, $\{T^n x\}$ has a limit u in M. To prove that $Tu = u$, let us consider the following inequalities.

$$\begin{aligned} d(u,\ Tu) &\leq d(u,\ T^{n+1}x) + d(TT^n x,\ Tu) \\ &\leq d(u,\ T^{n+1}x) + q \cdot \max\{d(T^n x,\ u),\ d(T^n x,\ T^{n+1}x); \\ &\qquad\qquad d(u,\ Tu);\ d(T^n x,\ Tu);\ d(T^{n+1}x,\ u)\} \\ &\leq d(u,\ T^{n+1}x) + q \cdot [d(T^n x,\ T^{n+1}x) + d(T^n x,\ u) \\ &\qquad\qquad + d(u,\ Tu) + d(T^{n+1}x,\ u)]. \end{aligned}$$

Hence

$$d(u,\ Tu) \leq \frac{1}{1-q}[(1+q)d(u,\ T^{n+1}x) + q \cdot d(u,\ T^n x) + q \cdot d(T^n x,\ T^{n+1}x)].$$

Since $\lim_n T^n x = u$, this shows that $d(u,\ Tu) = 0$, i.e. u is a fixed point under T. The uniqueness follows from the quasi-contractivity of T. So we have proved (a) and (b), as x was arbitrary. Letting m tend to infinity in (1), we obtain the inequality (c).

This completes the proof of the theorem.

The next result readily follows from the above theorem.

Theorem 2. *Let T be a mapping of a metric space M into itself and let M be T-orbitally complete. If there exists a positive integer k such that the iteration T^k is a quasi-contraction, then*

(a′) *T has a unique fixed point u in M,*

(b′) $\lim_n T^n x = u$, *and*

(c′) $d(T^n x,\ u) \leq q^m a(x)/(1-q)$ *for every $x \in M$,*

where $a(x) = \max\{d(T^i x,\ T^{i+k}x): i = 0, 1, \cdots, k-1\}$ *and* $m = E(n/k)$ *is the greatest integer not exceeding n/k.*

Proof. Since T^k has a unique fixed point u and $T^k(Tu) = T(T^k u) = Tu$, it follows that $Tu = u$. Its uniqueness is obvious. To show (c′), let n be any integer. Then $n = m \cdot k + j$, $0 \leq j < k$, $m \geq 0$, and for every $x \in M$, $T^n x = (T^k)^m T^j x$. Since T^k is a quasi-contraction, it follows from part (c) of Theorem 1 that

$$d(T^n x, u) \leq \frac{q^m}{1-q} d(T^j x, T^k T^j x)$$

$$\leq \frac{q^m}{1-q} \max\{d(T^i x, T^k T^i x): i = 0, 1, \cdots, k-1\},$$

which proves (c′), and hence (b′). This completes the proof of the theorem.

Note that Theorem 2.5 (Theorem 2.6) of [2] is a special case of Theorem 1 (Theorem 2). The example following Definition 1 shows that Theorem 1 is more general than Theorem 2.5 of [2]. In that example M is T-orbitally complete and o is a fixed point under T.

3. **Multi-valued quasi-contractions.** We shall now recall some terminologies. Let (M, d) be a metric space and let A, B be any subsets of M. We denote $D(A, B) = \inf\{d(a, b): a \in A, b \in B\}$, $\rho(A, B) = \sup\{d(a, b): a \in A, b \in B\}$, $BN(M) = \{A: \emptyset \neq A \subset M \text{ and } \delta(A) < +\infty\}$. Let $F: M \to M$ be a point to set correspondence and let $x_0 \in M$. An *orbit* of F at x_0 is a sequence $\{x_n: x_n \in Fx_{n-1}, n = 1, 2, \cdots\}$. A space M is said to be *F-orbitally complete* iff every Cauchy sequence which is a subsequence of an orbit of F at x for some $x \in M$, converges in M. Among the results established in [4] was the following: if $F: M \to BN(M)$ satisfies

(C) $\rho(Fx, Fy) \leq q \cdot \max\{d(x, y);\ \rho(x, Fx);\ \rho(y, Fy);\ \tfrac{1}{2}[D(x, Fy) + D(y, Fx)]\}$

for some $q < 1$ and if M is F-orbitally complete, then F has a unique fixed point u with $Fu = \{u\}$ and for each $x_0 \in M$ there exists an orbit $\{x_n\}$ of F at x_0 such that $\lim_n x_n = u$. The following is an extension of the above statement.

Theorem 3. *Let $F: M \to BN(M)$ be a multi-valued mapping on a metric space M and let M be F-orbitally complete. If F satisfies*

(D) $\rho(Fx, Fy) \leq q \cdot \max\{d(x, y);\ \rho(x, Fx);\ \rho(y, Fy);\ D(x, Fy);\ D(y, Fx)\}$

for some $q < 1$ and all $x, y \in M$, then

(i) *F has a unique fixed point* u *in M and $Fu = \{u\}$,*

(ii) *for each $x_0 \in M$ there exists an orbit $\{x_n\}$ of F at x_0 such that $\lim_n x_n = u$, and*

(iii) $d(x_n, u) \leq ((q^{1-a})^n/(1 - q^{1-a}))d(x_0, x_1)$,

where $a < 1$ is any fixed positive number.

Proof. Let $a \in (0, 1)$ be any number. Define a single-valued mapping $T: M \to M$ as follows: for each $x \in M$ let Tx be a point of Fx, which satisfies

$d(x, Tx) \geq q^a \cdot \rho(x, Fx)$. A mapping T is then a quasi-contraction with $q_1 = q^{1-a}$. Indeed, for every $x, y \in M$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \rho(Fx, Fy) \\ &\leq q \cdot q^{-a} \max\{q^a d(x, y);\ q^a \rho(x, Fx);\ q^a \rho(y, Fy); \\ &\qquad\qquad q^a D(x, Fy);\ q^a D(y, Fx)\} \\ &\leq q^{1-a} \max\{d(x, y);\ d(x, Tx);\ d(y, Ty);\ d(x, Ty);\ d(y, Tx)\}, \end{aligned}$$

which means that T is a quasi-contraction. Clearly $u = Tu$ implies $u \in Fu$. Since F satisfies (D), $u \in Fu$ implies $\rho(Fu, Fu) \leq q \cdot \rho(u, Fu)$. This may happen only if $Fu = \{u\}$. Therefore, $u \in M$ is a fixed point of T iff **u** is a fixed point of F. Since for each $x \in M$ the sequence $\{T^n x\}$ is an orbit of F at x, the statements of Theorem 3 follow from Theorem 1.

REFERENCES

1. V. W. Bryant, *A remark on a fixed point theorem for iterated mappings*, Amer. Math. Monthly 75 (1968), 399–400. MR 37 #2210.

2. Lj. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. 12 (26) (1971), 19–26. MR 46 #8203.

3. ———, *Fixed point theorems for mappings with a generalized contractive iterate at a point*, Publ. Inst. Math. 13 (27) (1972), 11–16.

4. ———, *Fixed points for generalized multi-valued contractions*, Mat. Vesnik 9 (24) (1972), 265–272.

5. ———, *On contraction type mappings*, Math. Balkanica 1 (1971), 52–57.

6. L. Dube and S. Singh, *On multi-valued contraction mappings*, Bull. Math. Soc. Math. RSR 14 (1970), 307–310 (1971).

7. M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. 12 (1961), 7–10. MR 22 #11375.

8. L. F. Guseman, Jr., *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. 26 (1970), 615–618. MR 42 #919.

9. R. Kannan, *Some results on fixed points*. II, Amer. Math. Monthly 76 (1969), 405–408. MR 41 #2487.

10. S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969), 475–488. MR 40 #8035.

11. S. Reich, *Fixed points of contractive functions*, Bol. Un. Mat. Ital. (4) 5 (1972), 26–42.

12. V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. 23 (1969), 631–634. MR 40 #3531.

13. R. Smithson, *Fixed points for contractive multifunctions*, Proc. Amer. Math. Soc. 27 (1971), 192–194. MR 42 #2466.

MATEMATIČKI INSTITUT, BEOGRAD, YUGOSLAVIA