A GENERALIZATION OF BANACH'S CONTRACTION PRINCIPLE

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ABSTRACT. Let $T: M \to M$ be a mapping of a metric space $(M, d)$ into itself. A mapping $T$ will be called a quasi-contraction iff $d(Tx, Ty) \leq q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$ for some $q < 1$ and all $x, y \in M$. In the present paper the mappings of this kind are investigated. The results presented here show that the condition of quasi-contraction implies all conclusions of Banach's contraction principle. Multi-valued quasi-contractions are also discussed.

1. Introduction. The well-known Banach's contraction mapping principle states that if $T: M \to M$ is a contraction on $M$ (i.e. $d(Tx, Ty) \leq q \cdot d(x, y)$ for some $q < 1$ and all $x, y \in M$) and $M$ is complete, then

$(10)$ $T$ has a unique fixed point $u$ in $M,$
$(20)$ $\lim_{n \to \infty} T^n x = u,$ and
$(30)$ $d(T^n x, u) \leq q^n (1 - q)^{-1} d(x, Tx)$ for every $x \in M.$

A number of generalizations of this result have appeared [1], [2], [3], [7], [8], [9], [12]. In [2] we considered generalized contractions, defined as follows.

A mapping $T: M \to M$ is said to be a generalized contraction iff for every $x, y \in M$ there exist nonnegative numbers $q$, $r$, $s$ and $t$, which may depend on both $x$ and $y$, such that $\sup\{q + r + s + 2t: x, y \in M\} < 1$ and

\[ d(Tx, Ty) \leq q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)]. \]

(A)

S. Nadler [10] has extended Banach's contraction principle to multi-valued contractions. Many extensions of Nadler's result have been derived in recent years [4], [6], [11], [13]. In [4] we proved some fixed-point theorems for a class of multi-valued generalized contractions—the maps which include the single-valued generalized contractions.

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The purpose of this paper is to extend some results concerning generalized contractions of [2] and [4] to quasi-contractions. In §2 fixed-point theorems for single-valued quasi-contractions are proved and an example is given to show that the results established here are indeed extensions. In §3 it is shown that for multi-valued quasi-contractions a similar result is valid.

2. Quasi-contractions. Let $T$ be a mapping of a metric space $M$ into itself. For $A \subset M$ let $\delta(A) = \sup \{d(a, b) : a, b \in A\}$ and for each $x \in M$, let

$$O(x, n) = \{x, Tx, \ldots, T^n x\}, \quad n = 1, 2, \ldots,$$

$$O(x, \infty) = \{x, Tx, \ldots\}.$$

A space $M$ is said to be $T$-orbitally complete iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in M$ converges in $M$ (cf. [5]).

Definition 1. A mapping $T : M \to M$ of a metric space $M$ into itself is said to be a quasi-contraction iff there exists a number $q, 0 < q < 1$, such that

$$(B) \quad d(Tx, Ty) \leq q \cdot \max \{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$$

holds for every $x, y \in M$.

It is clear that condition (A) implies (B). The following example shows that a quasi-contraction need not be a generalized contraction.

Example. Let

$$M_1 = \{m/n : m = 0, 1, 3, 9, \ldots ; n = 1, 4, \ldots, 3k + 1, \ldots\},$$

$$M_2 = \{m/n : m = 1, 3, 9, 27, \ldots ; n = 2, 5, \ldots, 3k + 2, \ldots\},$$

and let $M = M_1 \cup M_2$ with the usual metric. Define $T : M \to M$ by

$$Tx = 3x/5, \quad \text{for } x \in M_1,$$

$$= x/8, \quad \text{for } x \in M_2.$$

The mapping $T$ is a quasi-contraction with $q = 3/5$. Indeed, if both $x$ and $y$ are in $M_1$ or in $M_2$, then $d(Tx, Ty) \leq (3/5)d(x, y)$. Now let $x$ be, for example, in $M_1$ and $y$ in $M_2$. Then

$$x > \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5} \left( x - \frac{5}{24} y \right) \leq \frac{3}{5} \left( x - \frac{1}{8} y \right) = \frac{3}{5} d(x, Ty);$$

$$x < \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5} \left( \frac{5}{24} y - x \right) \leq \frac{3}{5} (y - x) = \frac{3}{5} d(x, y).$$

Therefore, $T$ on $M$ satisfies the condition
\[ d(Tx, Ty) \leq (3/5) \max \{ d(x, y); d(x, Ty); d(y, Tx) \} \]

and hence (B).

To show that \( T \) is not a generalized contraction on \( M \), let \( x = 1 \) and \( y = \frac{1}{2} \). Then we have

\[
q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot (d(x, Ty) + d(y, Tx))
\]

\[
= q \cdot \frac{1}{2} + r \cdot \frac{2}{5} + s \cdot \frac{7}{15} + t \cdot \frac{83}{80}
\]

\[
< (q + r + s + 2t) \cdot \frac{83}{160} < \frac{83}{160} < \frac{43}{80} = d(Tx, Ty),
\]

as \( q + r + s + 2t < 1 \), and we see that condition (A) is not satisfied.

Before stating the fixed-point theorem for quasi-contractions we shall prove two lemmas on these mappings. The first of these lemmas is fundamental.

**Lemma 1.** Let \( T \) be a quasi-contraction on \( M \) and let \( n \) be any positive integer. Then for each \( x \in M \) and all positive integers \( i \) and \( j \), \( i, j \in \{1, 2, \ldots, n\} \) implies \( d(T^i x, T^j x) \leq q \cdot \delta[O(x, n)] \).

**Proof.** Let \( x \in M \) be arbitrary, let \( n \) be any positive integer and let \( i \) and \( j \) satisfy the condition of Lemma 1. Then \( T^{i-1} x, T^i x, T^{j-1} x, T^j x \in O(x, n) \) (where it is understood that \( T^0 x = x \)) and since \( T \) is a quasi-contraction, we have

\[
d(T^i x, T^j x) = d(T T^{i-1} x, T T^{j-1} x)
\]

\[
\leq q \cdot \max \{ d(T^{i-1} x, T^{j-1} x); d(T^{i-1} x, T^i x); d(T^{j-1} x, T^j x); \\
\quad d(T^{i-1} x, T^j x); d(T^i x, T^{j-1} x) \}
\]

\[
\leq q \cdot \delta[O(x, n)],
\]

which proves the lemma.

**Remark.** From this lemma it follows that if \( T \) is a quasi-contraction and \( x \in M \), then for every positive integer \( n \) there exists a positive integer \( k \leq n \), such that \( d(x, T^k x) = \delta[O(x, n)] \).

**Lemma 2.** If \( T \) is a quasi-contraction on \( M \), then

\[
\delta[O(x, \infty)] \leq (1/(1 - q)) d(x, Tx)
\]

holds for all \( x \in M \).

**Proof.** Let \( x \in M \) be arbitrary. Since \( \delta[O(x, 1)] \leq \delta[O(x, 2)] \leq \cdots \), we
see that $\delta(O(x, \infty)) = \sup \{\delta(O(x, n)) : n \in \mathbb{N}\}$. The lemma will follow if we show that $\delta(O(x, n)) \leq (1/(1 - q))d(x, Tx)$ for all $n \in \mathbb{N}$.

Let $n$ be any positive integer. From the remark to the previous lemma, there exists $T^k x \in O(x, n)$ ($1 \leq k \leq n$) such that $d(x, T^k x) = \delta(O(x, n))$. Applying a triangle inequality and Lemma 1, we get

$$d(x, T^k x) \leq d(x, Tx) + d(Tx, T^k x) \leq d(x, Tx) + q \cdot \delta(O(x, n))$$

$$= d(x, Tx) + q \cdot d(x, T^k x).$$

Therefore, $\delta(O(x, n)) = d(x, T^k x) \leq (1/(1 - q))d(x, Tx).$ Since $n$ was arbitrary, the proof is completed.

Now we can state our main result.

**Theorem 1.** Let $T$ be a quasi-contraction on a metric space $M$ and let $M$ be $T$-orbitally complete. Then

(a) $T$ has a unique fixed point $u$ in $M$,

(b) $\lim_n T^n x = u$, and

(c) $d(T^n x, u) \leq (q^n/(1 - q))d(x, Tx)$ for every $x \in M$.

**Proof.** Let $x$ be an arbitrary point of $M$. We shall show that the sequence of iterates $\{T^n x\}$ is a Cauchy sequence.

Let $n$ and $m$ ($n < m$) be any positive integers. Since $T$ is a quasi-contraction, it follows from Lemma 1 that

$$d(T^n x, T^m x) = d(TT^{n-1} x, T^{m-n+1} T^{n-1} x) \leq q \cdot \delta(O(T^{n-1} x, m - n + 1)).$$

According to the remark to Lemma 1, there exists an integer $k_1$, $1 \leq k_1 \leq m - n + 1$, such that

$$\delta(O(T^{n-1} x, m - n + 1)) = d(T^{n-1} x, T^{k_1} T^{n-1} x).$$

Again, by Lemma 1, we have

$$d(T^{n-1} x, T^{k_1} T^{n-1} x) = d(TT^{n-2} x, T^{k_1+1} T^{n-2} x)$$

$$\leq q \cdot \delta(O(T^{n-2} x, k_1 + 1))$$

$$\leq q \cdot \delta(O(T^{n-2} x, m - n + 2)).$$

Therefore, we have the following system of inequalities.

$$d(T^n x, T^m x) \leq q \cdot \delta(O(T^{n-1} x, m - n + 1)) \leq q^2 \cdot \delta(O(T^{n-2} x, m - n + 2)).$$

Proceeding in this manner, we obtain
Then it follows from Lemma 2 that

\[ d(T^n x, T^m x) \leq (q^n/(1 - q))d(x, Tx). \]

Since \( \lim_n q^n = 0 \), \( \{T^n x\} \) is a Cauchy sequence.

Again, \( M \) being \( T \)-orbitally complete, \( \{T^n x\} \) has a limit \( u \) in \( M \). To prove that \( Tu = u \), let us consider the following inequalities.

\[ d(u, Tu) \leq d(u, T^{n+1} x) + d(T^{n+1} x, Tu) \]
\[ \leq d(u, T^{n+1} x) + q \cdot \max\{d(T^n x, u), d(T^n x, T^{n+1} x); d(u, Tu); d(T^n x, Tu); d(T^{n+1} x, u)\} \]
\[ \leq d(u, T^{n+1} x) + q \cdot [d(T^n x, T^{n+1} x) + d(T^n x, u) \]
\[ + d(u, Tu) + d(T^{n+1} x, u)]. \]

Hence

\[ d(u, Tu) \leq \frac{1}{1 - q} [(1 + q)d(u, T^{n+1} x) + q \cdot d(u, T^n x) + q \cdot d(T^n x, T^{n+1} x)]. \]

Since \( \lim_n T^n x = u \), this shows that \( d(u, Tu) = 0 \), i.e. \( u \) is a fixed point under \( T \). The uniqueness follows from the quasi-contractivity of \( T \). So we have proved (a) and (b), as \( x \) was arbitrary. Letting \( m \) tend to infinity in (1), we obtain the inequality (c).

This completes the proof of the theorem.

The next result readily follows from the above theorem.

**Theorem 2.** Let \( T \) be a mapping of a metric space \( M \) into itself and let \( M \) be \( T \)-orbitally complete. If there exists a positive integer \( k \) such that the iteration \( T^k \) is a quasi-contraction, then

(a') \( T \) has a unique fixed point \( u \) in \( M \),

(b') \( \lim_n T^n x = u \), and

(c') \( d(T^n x, u) \leq q^m a(x)/(1 - q) \) for every \( x \in M \),

where \( a(x) = \max\{d(T^i x, T^{i+k} x); i = 0, 1, \ldots, k - 1\} \) and \( m = E(n/k) \) is the greatest integer not exceeding \( n/k \).

**Proof.** Since \( T^k \) has a unique fixed point \( u \) and \( T^k(Tu) = T(T^k u) = Tu \), it follows that \( Tu = u \). Its uniqueness is obvious. To show (c'), let \( n \) be any integer. Then \( n = m \cdot k + j \), \( 0 \leq j < k \), \( m \geq 0 \), and for every \( x \in M \), \( T^n x = (T^k)^m T^j x \). Since \( T^k \) is a quasi-contraction, it follows from part (c) of Theorem 1 that
\[ d(T^n x, u) \leq \frac{q^m}{1 - q} d(T^i x, \ T^k T^i x) \]
\[ \leq \frac{q^m}{1 - q} \max\{d(T^i x, \ T^k T^i x): i = 0, 1, \cdots, k - 1\}, \]
which proves \((c')\), and hence \((b')\). This completes the proof of the theorem.

Note that Theorem 2.5 (Theorem 2.6) of [2] is a special case of Theorem 1 (Theorem 2). The example following Definition 1 shows that Theorem 1 is more general than Theorem 2.5 of [2]. In that example \(M\) is \(T\)-orbitally complete and \(u\) is a fixed point under \(T\).

3. Multi-valued quasi-contractions. We shall now recall some terminologies. Let \((M, d)\) be a metric space and let \(A, B\) be any subsets of \(M\). We denote \(D(A, B) = \inf\{d(a, b): a \in A, \ b \in B\}\), \(\rho(A, B) = \sup\{d(a, b): a \in A, \ b \in B\}\), \(BN(M) = \{A: \emptyset \neq A \subset M \ and \ \delta(A) < +\infty\}\). Let \(F: M \rightarrow M\) be a point to set correspondence and let \(x_0 \in M\). An orbit of \(F\) at \(x_0\) is a sequence \(\{x_n: x_n \in Fx_{n-1}, n = 1, 2, \cdots\}\). A space \(M\) is said to be \(F\)-orbitally complete iff every Cauchy sequence which is a subsequence of an orbit of \(F\) at \(x\) for some \(x \in M\), converges in \(M\). Among the results established in [4] was the following: if \(F: M \rightarrow BN(M)\) satisfies

\[ (C) \ \rho(Fx, Fy) \leq q \cdot \max\{d(x, y); \ \rho(x, Fx); \ \rho(y, Fy); D(x, y); D(y, Fx)\} \]
for some \(q < 1\) and if \(M\) is \(F\)-orbitally complete, then \(F\) has a unique fixed point \(u\) with \(Fu = \{u\}\) and for each \(x_0 \in M\) there exists an orbit \(\{x_n\}\) of \(F\) at \(x_0\) such that \(\lim x_n = u\). The following is an extension of the above statement.

**Theorem 3.** Let \(F: M \rightarrow BN(M)\) be a multi-valued mapping on a metric space \(M\) and let \(M\) be \(F\)-orbitally complete. If \(F\) satisfies

\[ (D) \ \rho(Fx, Fy) \leq q \cdot \max\{d(x, y); \ \rho(x, Fx); \ \rho(y, Fy); D(x, y); D(y, Fx)\} \]
for some \(q < 1\) and all \(x, y \in M\), then

(i) \(F\) has a unique fixed point \(u\) in \(M\) and \(Fu = \{u\}\),

(ii) for each \(x_0 \in M\) there exists an orbit \(\{x_n\}\) of \(F\) at \(x_0\) such that

\[ \lim x_n = u, \] and

(iii) \(d(x_n, u) \leq ((q^{1-a}n/(1 - q^{1-a}))d(x_0, x_1), \)
where \(a < 1\) is any fixed positive number.

**Proof.** Let \(a \in (0, 1)\) be any number. Define a single-valued mapping \(T: M \rightarrow M\) as follows: for each \(x \in M\) let \(Tx\) be a point of \(Fx\), which satisfies
\[ d(x, Tx) \geq q^{a} \cdot \rho(x, Fx) \] A mapping \( T \) is then a quasi-contraction with \( q_1 = q^{1-a} \). Indeed, for every \( x, y \in M \) we have
\[
d(Tx, Ty) \leq \rho(Fx, Fy)
\]
\[
\leq q \cdot q^{-a} \max \{ q^a d(x, y); q^a \rho(x, Fx); q^a \rho(y, Fy); q^a D(x, Fy); q^a D(y, Fx) \}
\]
\[
\leq q^{1-a} \max \{ d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx) \},
\]
which means that \( T \) is a quasi-contraction. Clearly \( u = Tu \) implies \( u \in Fu \).
Since \( F \) satisfies (D), \( u \in Fu \) implies \( \rho(Fu, Fu) \leq q \cdot \rho(u, Fu) \). This may happen only if \( Fu = \{ u \} \). Therefore, \( u \in M \) is a fixed point of \( T \) iff \( u \) is a fixed point of \( F \). Since for each \( x \in M \) the sequence \( \{ T^n x \} \) is an orbit of \( F \) at \( x \), the statements of Theorem 3 follow from Theorem 1.

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