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SUMMABILITY METHODS FOR INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

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ABSTRACT. In this paper, we present certain theorems concerning the Cesaro (C, α), Abel (A), Euler (E, q) and Borel (B) summability of ΣY_i, where $Y_i = X_i - X_{i-1}$, $X_0 = 0$ and $X_1, X_2, \cdots$ are i.i.d. random variables. While the Kolmogorov strong law of large numbers and the Hartman-Wintner law of the iterated logarithm are related to $(C, 1)$ summability and involve the finiteness of, respectively, the first and second moments of X_1, their analogues for Euler and Borel summability involve different moment conditions, and the analogues for (C, α) and Abel summability remain essentially the same.

Let $X_1, X_2, \cdots$ be independent, identically distributed (i.i.d.) random variables. Let $Y_n = X_n - X_{n-1}$ $(X_0 = Y_0 = 0)$. Kolmogorov's strong law of large numbers asserts that $EX_1 = \mu$ iff ΣY_i is a.e. $(C, 1)$ summable to μ, i.e., the $(C, 1)$ limit of X_n is μ a.e. By the well-known inclusion theorems involving Cesaro and Abel summability (cf. [5, Theorems 43 and 55]), this implies that ΣY_i is a.e. (C, α) summable to μ for any $\alpha \geq 1$ and that ΣY_i is a.e. (A) summable to μ. In fact, the converse also holds in the present case, and we have the following theorem.

Theorem 1. *If $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables and $\alpha \geq 1$ and μ are given real numbers, then the following statements are equivalent:*

(1) $EX_1 = \mu$.

(2) $X_n \to \mu(C, 1)$ *a.e., i.e.,* $\lim_{n\to\infty} (1/n)(X_1 + \cdots + X_n) = \mu$ *a.e.*

(3) $X_n \to \mu(C, \alpha)$ *a.e., i.e.,* $\lim_{n\to\infty} \Sigma_{i=1}^{n-1} \binom{i+\alpha-1}{i} X_{n-i} / \binom{n+\alpha}{n} = \mu$ *a.e., where* $\binom{j+\beta}{j} = (\beta + 1) \cdots (\beta + j)/(j!)$.

(4) $X_n \to \mu(A)$ *a.e., i.e.,* $\lim_{\lambda\to 1-} (1 - \lambda) \Sigma_{i=1}^{\infty} \lambda^i X_i = \mu$ *a.e.*

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Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are well known (cf. [5, pp. 96, 100, 108]). We shall now prove that (4) $\Rightarrow$ (1). By (4),

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{\infty} e^{-n/m} X_n^s = 0 \quad \text{a.e.},$$

where $X_n^s = X_n - X_n'$ with X_n', $n \geq 1$, and X_n, $n \geq 1$, being i.i.d. Let

$$Y_m = \frac{1}{m} \sum_{n=1}^{m} e^{-n/m} X_n^s, \qquad Z_m = \frac{1}{m} \sum_{n=m+1}^{\infty} e^{-n/m} X_n^s.$$

Then $Y_m + Z_m \xrightarrow{P} 0$, Y_m and Z_m are independent and symmetric. Therefore it follows easily from the Lévy inequality [8, p. 247] that $Z_m \xrightarrow{P} 0$. Since Z_m and $(Y_1, \cdots, Y_m)$ are independent and $Y_m + Z_m \to 0$ a.e., $Z_m \xrightarrow{P} 0$, we obtain by Lemma 3 of [2] that $Y_m \to 0$ a.e. Letting $Y_m^{(1)} = Y_m - (em)^{-1} X_m^s$, since $(em)^{-1} X_m^s \xrightarrow{P} 0$, we again obtain by Lemma 3 of [2] that $X_m^s/m \to 0$ a.e. By the Borel-Cantelli lemma, this implies that $E|X_1| < \infty$. As established before, we then have $X_n \to EX_1(A)$ and so by (4), $\mu = EX_1$.

The classical law of the iterated logarithm gives us the rate at which the convergence in (2) takes place. Gaposhkin [4] has established the law of the iterated logarithm for the (C, α) and Abel methods in the case where X_1 is bounded. Actually by using either the Hartman-Wintner truncation scheme [6] or Strassen's strong invariance principle [9], we can extend Gaposhkin's results to the case where $0 < EX_1^2 < \infty$. The extension is the sharpest possible in the sense that its converse also holds, and this converse can be established by a modification of Feller's argument in [3]. The details of the proof are omitted here, and we simply state the results in the following theorem.

Theorem 2. *With the same notation as in Theorem* 1, *for any given positive number* σ, *the following statements are equivalent*:

(5) $EX_1 = 0$, $EX_1^2 = \sigma^2$.

(6) $\limsup_{n\to\infty} |\Sigma_1^n X_i| / \{2n \log\log n\}^{1/2} = \sigma$ *a.e.*

(7) $\limsup_{n\to\infty} |\Sigma_{i=1}^n \binom{i+\alpha-1}{i} X_{n-i}| / \{2n^{2\alpha-1} \log\log n\}^{1/2} = \sigma/\{(2\alpha-1)^{1/2}\Gamma(\alpha)\}$ *a.e.*

(8) $\limsup_{\lambda\to 1-} (1-\lambda)^{1/2} |\Sigma_{i=1}^{\infty} \lambda^i X_i| / \{\log|\log(1-\lambda)|\}^{1/2} = \sigma$ *a.e.*

Our primary interest in this paper lies in the analogue of the law of the iterated logarithm for the Euler and Borel methods of summation (cf. [5, Chapters 8 and 9]). Chow [1] has shown that unlike the Cesaro and Abel methods which require $E|X_1| < \infty$ for summability, the Euler and Borel methods require $EX_1^2 < \infty$ for summability. Specifically, if $X_1, X_2, \cdots$ are i.i.d., then the following statements are equivalent:

(9) $EX_1 = \mu$, $EX_1^2 < \infty$.

(10) $X_n \to \mu(E, q)$ for some (or equivalently for every) $q > 0$, i.e.,

$$\lim_{n\to\infty} (q+1)^{-n} \sum_{i=1}^{n} \binom{n}{i} q^{n-i} X_i = \mu \quad \text{a.e.}$$

(11) $X_n \to \mu(B)$, i.e.,

$$\lim_{\lambda\to\infty} e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} X_i = \mu \quad \text{a.e.}$$

The following theorem gives the rate at which the convergence in (10) or (11) takes place. It is interesting to compare the result with Theorem 2.

Theorem 3. *Suppose $q > 0$, $\sigma > 0$ and $X_1, X_2, \cdots$ are i.i.d. random variables. Then the following statements are equivalent:*

(12) $EX_1 = 0$, $EX_1^2 = \sigma^2$, $EX_1^4(\log^+|X_1| + 1)^{-2} < \infty$.

(13) $\limsup_{\lambda\to\infty} (\pi\lambda)^{1/4}(\log \lambda)^{-1/2} |\sum_{i=1}^{\infty} e^{-\lambda}(\lambda^i/i!)X_i| = \sigma/\sqrt{2}$ *a.e.*

(14) $\limsup_{n\to\infty} (\pi q n)^{1/4}(\log n)^{-1/2} |\sum_{i=1}^{n} (q+1)^{-n}\binom{n}{i} q^{n-i} X_i|$ $= \sigma(\tfrac{1}{2}(q+1))^{1/2}$ *a.e.*

Lemma. *Let $X_1, X_2, \cdots$ be independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2$ and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$. Suppose for $j \geq j_0$, there exists $r_j \geq 0$ such that $r_j = o(j^{1/4}(\log j)^{-1/2})$ and*

$$\exp\{t^2\sigma_j^2(1 - |t|r_j)/2\} \leq E\exp(tX_j) \leq \exp\{t^2\sigma_j^2(1 + |t|r_j/2)/2\} \tag{15}$$

whenever $|t|r_j \leq 1$. Then for any $c > 0$ and $\alpha > \tfrac{1}{2}$,

$$\limsup_{M\to\infty} \left(\frac{\pi M}{2c}\right)^{-1/4} (\log M)^{-1/2} \sum_{|m-M|\leq M^{\alpha}} X_m \exp\left(\frac{-c(m-M)^2}{M}\right) = \sigma \quad a.e. \tag{16}$$

Proof. To simplify the notation, we shall set $h = m - M$. Let $0 < \delta < \delta' < 1$ such that $(1-\delta')^2(1+\delta) < 1$. Choose $\zeta > 1$ such that

$$\delta > 3\left(2\int_{\zeta}^{\infty} e^{-2ct^2}\,dt\right)^{1/2} \text{ and } \int_{-\zeta}^{\zeta} e^{-2ct^2}\,dt > (1-\delta)^2(\pi/2c)^{1/2}.$$

Let

$$s_M^2 = \sum_{\zeta\sqrt{M}<|h|\le M^\alpha} (\exp(-2ch^2/M))\,\sigma_m^2 \sim 2\sigma^2\sqrt{M}\int_\zeta^\infty \exp(-2ct^2)\,dt,$$

$$Y_M = \left(\frac{1}{s_M}\right) \sum_{\zeta\sqrt{M}<|h|\le M^\alpha} (\exp(-ch^2/M))X_m.$$

Using (15) and Kolmogorov's (upper) exponential bounds (cf. [10, Lemma 1]), we obtain that for all large M,

$$(17)\qquad P\left[\sum_{\zeta\sqrt{M}<|h|\le M^\alpha} (\exp(-ch^2/M))X_m \ge \delta\sigma M^{1/4}(\log M)^{1/2}\right] \le P[Y_M \ge 3(\log M)^{1/2}] \le \exp\{-(9/4)\log M\}.$$

Replacing X_m by $-X_m$ in (17) gives a similar inequality, and an application of the Borel-Cantelli lemma gives

$$(18)\qquad \limsup_{M\to\infty} M^{-1/4}(\log M)^{-1/2}\left|\sum_{\zeta\sqrt{M}<|h|\le M^\alpha} (\exp(-ch^2/M))X_m\right| \le \delta\sigma \quad \text{a.e.}$$

Choose $\eta > \zeta^2$ and set $m_k = [\eta k^2]$. Using (15) and Kolmogorov's (lower) exponential bounds (cf. [10, Lemma 1]), it can be shown that for all large k,

$$(19)\qquad P\left[\left(\frac{\pi m_k}{2c}\right)^{-1/4} \sum_{|h|\le\zeta\sqrt{m_k}} X_m \exp\left(\frac{-ch^2}{m_k}\right) \ge (1-\delta)\sigma(2\log k)^{1/2}\right] \ge \exp\{-(1+\delta)(1-\delta')^2 \log k\}.$$

Now the σ-fields $\mathcal{F}_k = \mathcal{B}(X_m : |m-m_k| \le \zeta\sqrt{m_k})$ are independent for all large k. Hence it follows from (19) and the Borel-Cantelli lemma that

$$(20)\qquad \limsup_{k\to\infty}\left(\frac{\pi m_k}{2c}\right)^{-1/4}(2\log k)^{-1/2} \sum_{|h|\le\zeta\sqrt{m_k}} X_m \exp\left(\frac{-ch^2}{m_k}\right) \ge (1-\delta)\sigma \quad a.e.$$

Since $2\log k \sim \log m_k$ and δ is arbitrary, we obtain from (18) and (20) that the lim sup in (16) is $\ge \sigma$ a.e.

To prove that the lim sup in (16) is $\le \sigma$ a.e., let $\eta_k = (\log k)^{-2}$, $M_k = [\eta_k k^2]$. An application of Kolmogorov's (upper) exponential bounds gives

$$(21)\ \sum P\left[\left(\frac{\pi M_k}{2c}\right)^{-1/4} \sum_{|h|\le\zeta\sqrt{M_k}} X_m \exp\left(\frac{-ch^2}{M_k}\right) \ge (1+\delta)\sigma(2\log k)^{1/2}\right] < \infty.$$

Obviously (21) still holds with X_m replaced by $-X_m$, and noting that $\log M_k \sim 2\log k$, we obtain that

(22)

$$\limsup_{k\to\infty}\left(\frac{\pi M_k}{2c}\right)^{-1/4}(\log M_k)^{-1/2}\left|\sum_{|h|\le\zeta\sqrt{M_k}} X_m \exp\left(\frac{-ch^2}{M_k}\right)\right| \le (1+\delta)\sigma \quad \text{a.e.}$$

We now assert that

$$(23)\qquad \limsup_{k\to\infty} M_k^{-1/4}(\log M_k)^{-1/2} \max_{M_k<M\le M_{k+1}} |U_{k,M}| \le 6\delta \quad \text{a.e.},$$

where we define, for $M_k < M \le M_{k+1}$,

$$U_{k,M} = \sum_{|m-M|\le\zeta\sqrt{M}} X_m \exp\left(\frac{-c(m-M)^2}{M}\right) - \sum_{|m-M_k|\le\zeta\sqrt{M_k}} X_m \exp\left(\frac{-c(m-M_k)^2}{M_k}\right);$$

$$U_{k,M}^{(1)} = \sum_{M_k-\zeta\sqrt{M_k}\le m< M-\zeta\sqrt{M}} X_m \exp\left(\frac{-c(m-M_k)^2}{M_k}\right);$$

$$U_{k,M}^{(2)} = \sum_{M-\zeta\sqrt{M}\le m<M_k} X_m \left\{\exp\left(\frac{-c(m-M_k)^2}{M_k}\right) - \exp\left(\frac{-c(m-M)^2}{M}\right)\right\};$$

$$U_{k,M}^{(3)} = \sum_{M_k\le m<M} X_m \exp\left(\frac{-c(m-M_k)^2}{M_k}\right);$$

$$U_{k,M}^{(4)} = \sum_{M_k\le m<M} X_m \exp\left(\frac{-c(m-M)^2}{M}\right);$$

$$U_{k,M}^{(5)} = \sum_{M\le m<M_k+\zeta\sqrt{M_k}} X_m \left\{\exp\left(\frac{-c(m-M)^2}{M}\right) - \exp\left(\frac{-c(m-M_k)^2}{M_k}\right)\right\};$$

$$U_{k,M}^{(6)} = \sum_{M_k+\zeta\sqrt{M_k}\le m\le M+\zeta\sqrt{M}} X_m \exp\left(\frac{-c(m-M)^2}{M}\right).$$

We note that if k is sufficiently large and $M_k < M \le M_{k+1}$, then $M_k - \zeta\sqrt{M_k} < M - \zeta\sqrt{M} < M_k < M < M_k + \zeta\sqrt{M_k} < M + \zeta\sqrt{M}$, and so $|U_{k,M}| \le \Sigma_{i=1}^{6}|U_{k,M}^{(i)}|$.

First consider $U^{(1)}_{k,M}$. Setting $t = (8\delta^{-2}\log k)^{1/2}$, we have from (15) that

$$P[U^{(1)}_{k,M} \geq \delta M_k^{1/4}(\log M_k)^{1/2}]$$

$$\leq \{\exp(-t\delta(\log M_k)^{1/2})\}E\exp\{tM_k^{-1/4}U^{(1)}_{k,M}\}$$

$$\leq k^{-4}\exp\left\{t^2M_k^{-1/2}\sum_{M_k-\zeta\sqrt{M_k}\leq m<M-\zeta\sqrt{M}}\sigma_m^2\exp\left(\frac{-2c(m-M_k)^2}{M_k}\right)\right\} \tag{24}$$

$$= O(k^{-4}).$$

Replacing X_m by $-X_m$ in the above argument, we then obtain from (24) that

$$P\left[\max_{M_k<M\leq M_{k+1}}|U^{(1)}_{k,M}| \geq \delta M_k^{1/4}(\log M_k)^{1/2}\right] = O(k^{-4}(M_{k+1}-M_k)) = O(k^{-3}).$$

Therefore by the Borel-Cantelli lemma, we have for $i = 1$,

$$\limsup_{k\to\infty} M_k^{-1/4}(\log M_k)^{-1/2}\max_{M_k<M\leq M_{k+1}}|U^{(i)}_{k,M}| \leq \delta \quad \text{a.e.} \tag{25}$$

By a similar argument, we can show that (25) also holds for $i = 2, 3, 4, 5, 6$. Hence (23) holds, and the desired conclusion follows from (18), (22) and (23).

Proof of Theorem 3. We first assume (12) and show that (13) and (14) both hold. Set $b_m(\lambda) = e^{-\lambda}\lambda^m/(m!)$, let $1/2 < \alpha < 2/3$ and let k be an integer such that $k(1-\alpha) > 1$. Given $\delta > 0$, we choose $\epsilon > 0$ such that $\epsilon k < \delta$. From (12), it follows that

$$\lim_{n\to\infty} n^{-1/4}(\log n)^{-1/2}X_n = 0 \quad \text{a.e.} \tag{26}$$

To prove (13), we set $M = [\lambda]$, $h = m - M$. Since there exists $\nu > 0$ such that $\sum_{|h|>M^\alpha} b_m(\lambda) = O(\exp(-\lambda^\nu))$ (cf. [5, p. 200]), it follows from (26) that

$$\lim_{\lambda\to\infty}\lambda^{1/4}(\log\lambda)^{-1/2}\sum_{|h|>M^\alpha} b_m(\lambda)|X_m| = 0 \quad \text{a.e.} \tag{27}$$

Furthermore we note that (cf. [5, p. 200])

$$\sum_{|h|\leq M^\alpha}\left|(2\pi M)^{-1/2}\exp\left(\frac{-h^2}{2M}\right) - b_m(\lambda)\right|$$

$$= \sum_{|h|\leq M^\alpha}\left\{(2\pi M)^{-1/2}\exp\left(\frac{-h^2}{2M}\right)\right\}\left\{O\left(\frac{(1+|h|)}{\lambda}\right) + O\left(\frac{|h|^3}{\lambda^2}\right)\right\} = O(\lambda^{-1/2}). \tag{28}$$

Hence it follows from (26) that

$$\lim_{\lambda\to\infty} \lambda^{1/4}(\log \lambda)^{-1/2}\left\{ \sum_{|h|\le M^{\alpha}} \left|(2\pi M)^{-1/2}\exp\left(\frac{-h^2}{2M}\right) - b_m(\lambda)\right| |X_m| \right\} = 0 \quad \text{a.e.} \tag{29}$$

Define

$$X'_n = X_n I_{[|X_n|\ge \epsilon n^{1/4}(\log n)^{1/2}]}, \qquad X''_n = X_n I_{[|X_n|\le n^{1/4}/(\log n)]}$$

and $X'''_n = X_n - X'_n - X''_n$. Let $Y_n = X''_n - EX''_n$. Then $EY_n = 0$, $EY_n^2 = \sigma_n^2 \to \sigma^2$, $|Y_n| \le 2n^{1/4}/(\log n) = o(n^{1/4}(\log n)^{-1/2})$ and therefore satisfies the hypothesis of the lemma (cf. [8, p. 255]). Hence

$$\limsup_{\lambda\to\infty} (\pi\lambda)^{-1/4}(\log \lambda)^{-1/2} \sum_{|h|\le M^{\alpha}} Y_m \exp\left(\frac{-h^2}{2M}\right) = \sigma \quad \text{a.e.} \tag{30}$$

Noting that $EX_1 = 0$, we have for $|h| \le M^{\alpha}$,

$$|EX''_m| = |EX_1 I_{[|X_1|> m^{1/4}/(\log m)]}| \le m^{-1/2}(\log m)^2 E|X_1|^3. \tag{31}$$

Hence

$$\lim_{\lambda\to\infty} \lambda^{1/4}(\log \lambda)^{-1/2} \sum_{|h|\le M^{\alpha}} \left(M^{-1/2}\exp\left(\frac{-h^2}{2M}\right)\right)|EX''_m| = 0.$$

Also by (26), $P[X'_n \ne 0 \text{ i.o.}] = 0$. Since $m^{1/4}/(\log m) < |X'''_m| < \epsilon m^{1/4}(\log m)^{1/2}$, we have for all large M,

$$P\left[\sum_{|h|\le M^{\alpha}} |X'''_m| \ge \delta M^{1/4}(\log M)^{1/2}\right] \le P\left[\sum_{|h|\le M^{\alpha}} \frac{|X'''_m|}{(\epsilon m^{1/4}(\log m)^{1/2})} \ge k\right]$$

$$\le P[X'''_m \ne 0 \text{ for at least } k \text{ of the indices } M - M^{\alpha} \le m \le M + M^{\alpha}]$$

$$\le \binom{1 + 2[M^{\alpha}]}{k} P^k[|X_1|^4 > (M - M^{\alpha})\{\log(M - M^{\alpha})\}^{-4}]$$

$$= O(M^{\alpha k - k}(\log M)^{6k}), \quad \text{by the Markov inequality.}$$

Since $k - \alpha k > 1$, an application of the Borel-Cantelli lemma then gives

$$\limsup_{\lambda\to\infty} \lambda^{1/4}(\log \lambda)^{-1/2} \sum_{|h|\le M^{\alpha}} M^{-1/2}|X'''_m| \le \delta \quad \text{a.e.} \tag{32}$$

Hence we have proved (13). To prove (14), setting $M = [(n+1)/(q+1)]$, $c = (q+1)/2q$, we can use a similar argument as before (cf. [5, p. 201]).

We now prove that (13) implies (12). From (13), it is clear that $X_n \to 0(B)$ a.e. By the equivalence of (9) and (11), we have $EX_1 = 0$ and $EX_1^2 < \infty$. For $n = 3, 4, \cdots$, let $d_m(n) = (\pi n)^{1/4}(\log n)^{-1/2}b_m(n)$, $Y_n = \sum_{m=1}^n d_m(n)X_m$, $Z_n = \sum_{m=n+1}^\infty d_m(n)X_m$. Since $EX_1 = 0$ and $EX_1^2 < \infty$, it follows from Tchebychev's inequality that $Z_n \xrightarrow{P} 0$. Now Z_n is independent of $(Y_1, \cdots, Y_n)$ and $\limsup_{n \to \infty} |Y_n + Z_n| = \sigma/\sqrt{2}$ a.e. Hence by Lemma 1 of [7], $\limsup_{n\to\infty} |Y_n| \leq \sigma/\sqrt{2}$ a.e. But $Y_n = \sum_{m=1}^{n-1} d_m(n)X_m + d_n(n)X_n$ and $d_n(n)X_n \xrightarrow{P} 0$. Therefore applying Lemma 1 of [7] again, we have

$$\limsup_{n\to\infty} \left| \sum_{m=1}^{n-1} d_m(n)X_m \right| \leq \frac{\sigma}{\sqrt{2}} \quad \text{a.e.}$$

and

$$\limsup_{n\to\infty} |d_n(n)X_n| \leq \sqrt{2}\sigma \quad \text{a.e.}$$

Hence

$$\limsup_{n\to\infty} n^{-1/4}(\log n)^{-1/2}|X_n| < \infty \quad \text{a.e.},$$

and since this lim sup can only be ∞ a.e. of 0 a.e., it follows that $n^{-1/4}(\log n)^{-1/2}X_n \to 0$ a.e. Therefore $EX_1^4(\log^+|X_1| + 1)^{-2} < \infty$. It is now obvious from (13) that $EX_1^2 = \sigma^2$. In a similar way, we can prove that (14) implies (12).

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