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# DIAGONAL EQUIVALENCE TO MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS. II

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ABSTRACT. Let $A$ be a nonnegative $m \times n$ matrix and let $r = (r_1, \cdots, r_m)$ and $c = (c_1, \cdots, c_n)$ be positive vectors such that $\Sigma_{i=1}^m r_i = \Sigma_{j=1}^n c_j$. It is well known that if there exists a nonnegative $m \times n$ matrix $B$ with the same zero pattern as $A$ having the $i$th row sum $r_i$ and $j$th column sum $c_j$, there exist diagonal matrices $D_1$ and $D_2$ with positive main diagonals such that $D_1 A D_2$ has $i$th row sum $r_i$ and $j$th column sum $c_j$. However the known proofs are at best cumbersome. It is shown here that this result can be obtained by considering the minimum of a certain real-valued function of $n$ positive variables.

It has been shown originally by Sinkhorn and Knopp [8] and Brualdi, Parter, and Schneider [3] that if $A$ is a nonnegative fully indecomposable matrix, i.e. $A$ contains no $s \times (n-s)$ zero submatrix, then there exists a doubly stochastic matrix of the form $D_1 A D_2$ where $D_1$ and $D_2$ are diagonal matrices with positive main diagonals. Later Djoković [4], and independently, London [5], proved the same theorem by considering the minimum of

$$f(x) = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\Big/\prod_{j=1}^{n} x_j \tag{1}$$

for vectors $x = (x_1, \cdots, x_n)$ with positive coordinates.

In the meantime Menon [6] had obtained the following modification of this result.

**Theorem 1.** *Let $A$ be a nonnegative $m \times n$ matrix and let $r = (r_1, \cdots, r_m)$ and $c = (c_1, \cdots, c_n)$ be positive vectors such that $\Sigma_{i=1}^m r_i = \Sigma_{j=1}^n c_j$. If there exists a nonnegative $m \times n$ matrix $B$ with the same zero pattern as $A$, i.e. $b_{ij} = 0 \Leftrightarrow a_{ij} = 0$, having ith row sum $r_i$ and jth column sum $c_j$, then*

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*there exist diagonal matrices $D_1$ and $D_2$ with positive main diagonals such that $D_1AD_2$ has ith row sum $r_i$ and jth column sum $c_j$.*

Brualdi [2] showed that the existence of $B$ in Theorem 1 is equivalent to the conditions that

(1) $A[E|F) = 0$, $A(E|F] \neq 0 \Rightarrow \sum_{i\in E} r_i < \sum_{j\in F} c_j$, and

(2) $A[E|F) = 0$, $A(E|F] = 0 \Rightarrow \sum_{i\in E} r_i = \sum_{j\in F} c_j$,

if $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$ holds.

The notation used has the following meaning. If $E$ is a proper nonvoid subset of $M = \{1, \cdots, m\}$ and $F$ is a proper nonvoid subset of $N = \{1, \cdots, n\}$, then $A[E|F)$ is that submatrix of $A$ obtained by deleting from $A$ those rows whose indices do not belong to $E$ and those columns whose indices belong to $F$. The rows and columns of this submatrix appear in the same order as in $A$: rows are counted from top to bottom; columns are counted from left to right. $A(E|F]$ is that submatrix of $A$ obtained by deleting from $A$ those rows whose indices belong to $E$ and those columns whose indices do not belong to $F$, where, as before, the rows and columns of this submatrix appear in the same order as in $A$. Observe that the submatrix $A(E|F]$ is the same as the submatrix $A[M - E|N - F)$. In the course of the paper two other submatrix notations are used. $A[E|F]$ is used to denote the submatrix $A[E|N - F) = A(M - E|F]$ in $A$; $A(E|F)$ is used to denote the submatrix $A[M - E|F) = A(E|N - F]$ in $A$.

Menon and Schneider [7] have given another proof of the Menon-Brualdi results.

It is the intent of this paper to show how the Djoković-London formula can be modified to yield the Menon-Brualdi-Schneider results.

We shall require the following lemma which follows at once from the concavity of the logarithm function. See [1, p. 7].

**Lemma.** *Let $x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_n$ be nonnegative real numbers and put $\lambda_1 + \cdots + \lambda_n = \lambda$. Then if $0^0$ is taken to be 1,*

$$\left(\sum_{k=1}^{n} \lambda_k x_k\right)^{\lambda} \geq \lambda^{\lambda}\left(\prod_{k=1}^{n} x_k^{\lambda_k}\right).$$

We now prove the intended result. We shall assume that whenever there is a submatrix $A[E|F) = 0$ in $A$, $A(E|F] \neq 0$, for otherwise we could establish the result for the submatrices $A[E|F]$ and $A(E|F)$. We assume that Brualdi's condition (1) holds and that $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$.

Put

$$\phi(x) = \prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j=1}^{n} x_j^{c_j},$$

where $x = (x_1, \cdots, x_n)$ is positive, i.e. $x \in (R^n)^+$. We shall consider the problem of determining the minimum of $\phi$ on $(R^n)^+$. Since $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$, $\phi(\lambda x) = \phi(x)$ for all $\lambda > 0$ and thus we can restrict our attention to the set $K$ of $x \in (R^n)^+$ for which $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2} = 1$.

Suppose on $K$, $x \longrightarrow \Delta$, the boundary of $(R^n)^+$. Let $F = \{j | x_j \longrightarrow 0\}$ and then set $E = \{i | a_{ij} = 0 \text{ for all } j \notin F\}$. Since $x \longrightarrow \Delta$ on $K$, $F$ is a nonvoid proper subset of $\{1, \cdots, n\}$. Since every $c_j > 0$, $E$ is a proper subset of $\{1, \cdots, m\}$. If $E = \varnothing$, $\phi(x) \longrightarrow \infty$ as $x \longrightarrow \Delta$. If $E \neq \varnothing$, we write $\phi(x) = \phi_1(x)\phi_2(x)$ where

$$\phi_1(x) = \prod_{i\in E}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\in F} x_j^{c_j} = \prod_{i\in E}\left(\sum_{j\in F} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\in F} x_j^{c_j} \tag{3}$$

and

$$\phi_2(x) = \prod_{i\notin E}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\notin F} x_j^{c_j}. \tag{4}$$

Since $\phi_2$ has a positive limit as $x \longrightarrow \Delta$, we concentrate on $\phi_1$.

Let $B$ be as in Theorem 1. Then $B[E|F) = 0$ and therefore $\sum_{j\in F} b_{ij} = r_i$ for each $i \in E$ and hence from the Lemma

$$\left(\sum_{j\in F} a_{ij}x_j\right)^{r_i} \geq r_i^{r_i}\frac{\prod_{j\in F} a_{ij}^{b_{ij}}}{\prod_{j\in F} b_{ij}^{b_{ij}}} \prod_{j\in F} x_j^{b_{ij}} = \theta_i \prod_{j\in F} x_j^{b_{ij}} \tag{5}$$

for all $i \in E$, where $0^0$ is taken to be 1. Whence

$$\phi_1(x) \geq \prod_{i\in E} \theta_i \Big/ \prod_{j\in F} x_j^{c_j - \Sigma_{i\in E} b_{ij}}. \tag{6}$$

Since $\sum_{i=1}^{m} b_{ij} = c_j$, $j = 1, \cdots, n$, certainly $\sum_{i\in E} b_{ij} \leq c_j$ for every $j \in F$. However since $\sum_{j\in F} c_j > \sum_{i\in E} r_i = \sum_{i\in E}\sum_{j\in F} b_{ij}$, $\sum_{i\in E} b_{ij_0} < c_{j_0}$ for at least one $j_0 \in F$. Thus $\phi_1(x) \longrightarrow \infty$ and so $\phi(x) \longrightarrow \infty$ as $x \longrightarrow \Delta$.

It follows that $\phi$ achieves a minimum on $(R^n)^+$. At such a point $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$, $\partial \ln \phi(x)/\partial x_k = 0$ for $k = 1, \cdots, n$. Whence

$$\sum_{i=1}^{m} r_i\left(a_{ik} \Big/ \sum_{j=1}^{n} a_{ij}\bar{x}_j\right) - c_k/\bar{x}_k = 0, \tag{7}$$

$k = 1, \cdots, n$. Put $\bar{y}_i = r_i / \sum_{j=1}^n a_{ij} \bar{x}_j$, $i = 1, \cdots, m$, and then set $D_1 = \mathrm{diag}(\bar{y}_1, \cdots, \bar{y}_m)$, $D_2 = \mathrm{diag}(\bar{x}_1, \cdots, \bar{x}_n)$. Then $D_1 A D_2$ satisfies the conclusion of Theorem 1.

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