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SUMMABILITY METHODS FAIL FOR THE 2^nTH PARTIAL SUMS OF FOURIER SERIES[1]

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ABSTRACT. Although the Fourier series of a continuous function need not converge everywhere, it was an important discovery of Fejér that this series must be Cesàro summable. Indeed, it is a frequent occurrence that convergence may be restored to an expansion by use of an appropriate summability method. What we show in this note is that the very opposite phenomenon can occur. Namely, that if one considers only the 2^nth partial sums of the Fourier series, there is no summability method *whatever* which produces convergence for all continuous functions.

We need only consider regular matrix summability methods since a "sequence-to-function" method may be viewed through a sequence of the continuous variable (divergence through this subsequence surely implies divergence through the continuous variable).

Theorem. *Let N denote the summability method which sends x_n into x_{2^n}. If M is any regular summability method, then $M \cdot N$ is not effective for Fourier series.*

Proof. Let M have the matrix m_{ij} so that $\sum_{j=1}^{\infty} |m_{ij}| \leq c$ for all i, $\lim_{i\to\infty} \sum_{j=1}^{\infty} m_{ij} = 1$, $\lim_{i\to\infty} m_{ij} = 0$ for all j. Our assertion is equivalent to the fact that $\int_0^{\pi} |\sum_{j=1}^{\infty} m_{ij} \sin 2^j x| (dx/x)$ is unbounded as $i \to \infty$ and we show that, indeed, it goes to ∞. Now our conditions on m_{ij} surely imply that $\lim_{i\to\infty} \sum_{j=1}^{\infty} j^2 m_{ij}^2 = \infty$. For we have

$$\sum_{j=1}^{\infty} j^2 m_{ij}^2 \geq \sum_{j>n} j^2 m_{ij}^2 \geq \left(\sum_{j>n} m_{ij}\right)^2 \left(\sum_{j>n} \frac{1}{j^2}\right)^{-1},$$

by the Schwarz inequality, and this is $\geq n(\sum_{j>n} m_{ij})^2$. Thus, for any fixed n,

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$$\varliminf_{i\to\infty} \sum_{j=1}^{\infty} j^2 m_{ij}^2 \geq n \lim_{i\to\infty}\left(\sum_{j>n} m_{ij}\right)^2 = n.$$

Our theorem therefore follows from the

Lemma. $\int_0^\pi |\Sigma\, a_j \sin 2^j x|\,(dx/x) \geq (1/5)(\Sigma\, j^2 a_j^2)^{1/2} - \pi \Sigma\, |a_j|$.

For proof, split the given integral into $\Sigma_{k=1}^\infty I_k$ with

$$I_k = \int_{\pi\cdot 2^{-k}}^{2\pi\cdot 2^{-k}} \left|\sum a_j \sin 2^j x\right| \frac{dx}{x}.$$

We have then

$$I_k = \int_\pi^{2\pi} \left|\sum a_j \sin 2^{j-k} x\right| \frac{dx}{x} \geq J_k - \delta_k$$

where

$$J_k = \int_\pi^{2\pi} \left|\sum_{j\geq k} a_j \sin 2^{j-k} x\right| \frac{dx}{x}, \qquad \delta_k = \int_\pi^{2\pi} \left|\sum_{j<k} a_j \sin 2^{j-k} x\right| \frac{dx}{x}.$$

Regarding δ_k we note that

$$\delta_k \leq \int_\pi^{2\pi} \sum_{j<k} |a_j| \cdot 2^{j-k} x \cdot \frac{dx}{x} = \pi \sum_{j<k} |a_j| 2^{j-k}$$

so that

$$\sum \delta_k \leq \sum_{k=1}^{\infty} \sum_{j<k} |a_j| 2^{j-k} = \pi \sum_j |a_j| \sum_{k>j} 2^{j-k} = \pi \sum |a_j| \cdot 1.$$

As for the J_k we have

$$2J_k \geq 2\int_\pi^{2\pi} \left|\sum_{j\geq k} a_j \sin 2^{j-k} t\right| dt \cdot \frac{1}{2\pi}$$

$$= \frac{1}{\pi}\int_0^\pi |a_k \sin t + a_{k+1} \sin 2t + a_{k+2} \sin 4t + \cdots|\, dt,$$

and this expression is the L^1 norm of a trigonometric gap series. For such series the L^1 and L^2 norms are equivalent [1]. Indeed in the present case we actually obtain $J_k \geq (1/5)(\Sigma_{j\geq k}\, a_j^2)^{1/2}$.

Our Lemma now follows from the following simple elementary inequality

$$\sum_{k=1}^{\infty}\left(\sum_{j\geq k} a_j^2\right)^{1/2} \geq \left(\sum_{j=1}^{\infty} j^2 a_j^2\right)^{1/2}.$$

Indeed, calling $R_k = (\Sigma_{j\geq k} a_j^2)^{1/2}$, we have, by the monotonicity of R_k

$$\left(\sum_{k=1}^{\infty} R_k\right)^2 = \sum_{k=1}^{\infty} R_k\left(2\sum_{i<k} R_i + R_k\right) \geq \sum_{k=1}^{\infty} R_k^2(2(k-1)R_k + R_k).$$

In turn this is equal to

$$\sum_{k=1}^{\infty} (2k-1)R_k^2 = \sum_{k=1}^{\infty} (2k-1) \sum_{j\geq k} a_j^2$$

$$= \sum_j a_j^2 \sum_{k\leq j} (2k-1) = \sum_j j^2 a_j^2$$

and so our proof is complete.

REFERENCE

1. R. E. Edwards, *Fourier series: A modern introduction*, Vol. II, Holt, Rinehart and Winston, New York, 1967, p. 220. MR 36 #5588.

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