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TRACE CLASS PERTURBATIONS OF ISOMETRIES AND UNITARY DILATIONS

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ABSTRACT. Let U and V be isometries acting on a Hilbert space. If $U - V$ belongs to the trace class we show that the absolutely continuous parts of the corresponding minimal unitary dilations are unitarily equivalent.

Introduction. With X and Y complex Hilbert spaces let $\mathcal{B}(X, Y)$ denote the collection of bounded linear operators from X to Y; the symbol $\mathcal{T}(X)$ will represent the trace class elements in $\mathcal{B}(X, X)$. If A is a (not necessarily bounded) selfadjoint operator in X with the spectral representation $A = \int_R \lambda\, dE_\lambda$ and $x \in X$, then $\mu_x(\Delta) = \|E(\Delta)x\|^2$ is a nonnegative, completely additive measure defined for Borel sets Δ. If this measure is absolutely continuous with respect to Lebesgue measure, we say that x is absolutely continuous with respect to A. The collection $X(A)_{\mathrm{ac}}$ of all vectors x for which the measure $\mu_x(\cdot)$ is absolutely continuous is a closed subspace which reduces A [4], and the restriction of A to this space, A_{ac}, is called the absolutely continuous part of A. A similar definition applies to a unitary operator A.

1. An important result on the perturbation of continuous spectra was given by Rosenblum [1] and Kato [2] and later generalized by Kuroda [3]. This theorem asserts that if $\{A, B\}$ is a pair of selfadjoint operators in a Hilbert space X such that $(A - z)^{-1} - (B - z)^{-1} \in \mathcal{T}(X)$ for some z with $\operatorname{Im} z \neq 0$, then A_{ac} is unitarily equivalent to B_{ac}. By taking Cayley transforms this may be rephrased in terms of unitary operators, i.e., if U and V are unitary operators for which $U - V \in \mathcal{T}(X)$, then U_{ac} and V_{ac} are unitarily equivalent. The object of this note is to prove an extension of this fact for pairs of isometries. Precisely, we prove the following:

Theorem. *Let U and V be isometries in X with minimal unitary dilations*

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U' and V', respectively. If $U - V \in \mathfrak{T}(X)$, then U'_{ac} and V'_{ac} are unitarily equivalent.

Proof. Since $U - V \in \mathfrak{T}(X)$, it is certainly compact; thus it suffices to consider the case of a separable space X. Again, since $U - V$ is compact, a property from index theory [5] says that $\text{index}(U) = \text{index}(V)$. But $\text{index}(I) = -\text{dimension } \mathcal{R}(I)^{\perp}$ ($\mathcal{R}(I) \equiv \text{range } I$), for any isometry I on X; hence $\text{dimension } \mathcal{R}(U)^{\perp} = \text{dimension } \mathcal{R}(V)^{\perp}$, (the symbol $\perp$ denotes orthogonal complement). Let α equal this common value. If α is infinite, then both U'_{ac} and V'_{ac} have spectral multiplicity uniformly $\aleph$ over the entire unit circle [6, Theorem 1, p. 271] and therefore U'_{ac} is unitarily equivalent to V'_{ac}. It remains to consider the case when α is a finite number.

Our strategy consists in extending U and V to minimal unitary dilations U' and V' on the same dilation space X' in such a manner that $U' - V' \in \mathfrak{T}(X')$. The assertion of the Theorem will then follow by invoking the result of Rosenblum-Kato-Kuroda for the pair $\{U', V'\}$.

The construction uses the Wold decomposition [6, Theorem 1.1, p. 3]: Let

$$U \cong S_{\alpha} \oplus U_u, \tag{1}$$

$$V \cong S_{\alpha} \oplus V_u \tag{2}$$

($\cong$ denotes unitary equivalence) be Wold decompositions for U and V so that S_{α} is a unilateral shift of multiplicity α, and $\{U_u, V_u\}$ are unitary operators not necessarily acting on the same space. Let h denote the l_2 space with dimension α. Without loss of generality we may assume that our original space X decomposes into an orthogonal sum $X_{\alpha} \oplus X_u(U)$ such that X_{α} is the Hilbert space whose elements are vector functions $\xi = \langle \xi_0, \xi_1, \xi_2, \cdots \rangle$ with $\xi_n \in h$, $\|\xi\|^2 = \Sigma_{n=0}^{\infty} \|\xi_n\|^2 < \infty$. The operator S_{α} acts on X_{α} by $S_{\alpha}\xi = \langle \eta_0, \eta_1, \eta_2, \cdots \rangle$; $\eta_0 = 0$, $\eta_n = \xi_{n-1}$, $n \geq 1$, and U_u operates in $X_u(U)$. Moreover, with V_u operating in the space $X_u(V)$, the decomposition (2) implies the existence of an isometric map ϕ of $Y \equiv X_{\alpha} \oplus X_u(V)$ onto X such that $V = \phi(S_{\alpha} \oplus V_u)\phi^*$, where $\phi^* \in \mathcal{B}(X, Y)$ is the adjoint map. Let us define dilation spaces by setting

$$X' = \hat{X}_{\alpha} \oplus' X \equiv \hat{X}_{\alpha} \oplus' X_{\alpha} \oplus X_u(U),$$

$$Y' = \hat{X}_{\alpha} \oplus' Y \equiv \hat{X}_{\alpha} \oplus' X_{\alpha} \oplus X_u(V)$$

where $\hat{X}_{\alpha}$ is the Hilbert space whose elements are the vectors $\xi = \langle \cdots, \xi_{-3}, \xi_{-2}, \xi_{-1} \rangle$ with $\xi_n \in h$, $\|\xi\|^2 = \Sigma_{n=1}^{\infty} \|\xi_{-n}\|^2 < \infty$, and $\oplus'$ denotes orthogonal direct sum. A minimal unitary dilation of U is then given by the operator $U' \in \mathfrak{B}(X', X')$ defined as follows [6, Chapter 1, §2]: If

$$x' = \langle \cdots, \xi_{-3}, \xi_{-2}, \xi_{-1} \rangle \oplus' \langle \xi_0, \xi_1, \xi_2, \cdots \rangle \oplus x_u,$$

set

$$U'x' = \langle \cdots, \xi_{-4}, \xi_{-3}, \xi_{-2} \rangle \oplus' \langle \xi_{-1}, \xi_0, \xi_1, \cdots \rangle \oplus U_u x_u.$$

Further, we construct a minimal unitary dilation V' of V by setting $V' = \phi' M' \phi'^*$ where $M' \in \mathfrak{B}(Y', Y')$; $\phi' \in \mathfrak{B}(Y', X')$, $\phi'^* \in \mathfrak{B}(X', Y')$ are defined by

$$M'y' = \langle \cdots, \xi_{-4}, \xi_{-3}, \xi_{-2} \rangle \oplus' \langle \xi_{-1}, \xi_0, \xi_1, \cdots \rangle \oplus V_u y_u,$$

$$\phi' y' = \langle \cdots, \xi_{-3}, \xi_{-2}, \xi_{-1} \rangle \oplus' \phi(\langle \xi_0, \xi_1, \xi_2, \cdots \rangle \oplus y_u)$$

for

$$y' = \langle \cdots, \xi_{-3}, \xi_{-2}, \xi_{-1} \rangle \oplus' \langle \xi_0, \xi_1, \xi_2, \cdots \rangle \oplus y_u$$

and ϕ'^* is the adjoint of ϕ'.

We shall now prove that $U' - V' \in \mathfrak{J}(X')$. For convenience we introduce the following operators: With

$$x' = \xi^{(1)} \oplus' \xi^{(2)} \oplus x \in X'; \qquad \xi^{(1)} = \langle \cdots, \xi_{-3}, \xi_{-2}, \xi_{-1} \rangle \in \hat{X}_\alpha,$$

put

$$Px' = \xi^{(2)} \oplus x \in X,$$

$$P_{-1}x' = 0 \oplus' \langle \xi_{-1}, 0, 0, \cdots \rangle \oplus 0 \in X',$$

$$Q_{-1}x' = 0 \oplus' \langle \xi_{-1}, 0, 0, \cdots \rangle \oplus 0 \in Y'$$

so that $P \in \mathfrak{B}(X', X)$, $P_{-1} \in \mathfrak{B}(X', X')$ and $Q_{-1} \in \mathfrak{B}(X', Y')$. Note that P is a projection, whereas P_{-1} and Q_{-1} are shifts followed by projections of rank α; in particular, $P_{-1} \in \mathfrak{J}(X')$. We also introduce the injection operator of X into X' by setting $Ix = 0 \oplus' x \in X'$ so $I \in \mathfrak{B}(X, X')$.

With these definitions we shall calculate that

$$U' - V' = P_{-1} + I(U - V)P - \phi'^* Q_{-1}.$$

Choose $x' = \xi^{(1)} \oplus' \xi^{(2)} \oplus x$ as above. We then have

$$\begin{aligned}(U' - V')x' &= U'x' - V'x' = U'x' - \phi' M' \phi'^* x' \\ &= \eta^{(1)} \oplus' \eta^{(2)} \oplus U_u x - \phi' M'[\xi^{(1)} \oplus' \phi^*(\xi^{(2)} \oplus x)]\end{aligned}$$

where

$$\eta^{(1)} = \langle \cdots, \xi_{-4}, \xi_{-3}, \xi_{-2} \rangle, \qquad \eta^{(2)} = \langle \xi_{-1}, \xi_0, \xi_1, \cdots \rangle.$$

Observe that

$$\eta^{(1)} \oplus' \eta^{(2)} \oplus U_u x = \eta^{(1)} \oplus' \tau^{(1)} \oplus 0 + 0 \oplus' U(\xi^{(2)} \oplus x)$$

where $\tau^{(1)} = \langle \xi_{-1}, 0, 0, \cdots \rangle$. Also,

$$\phi' M'[\xi^{(1)} \oplus' \phi^*(\xi^{(2)} \oplus x)] = \phi'[\eta^{(1)} \oplus' 0 \oplus 0 + 0 \oplus' \tau^{(1)} \oplus 0] + \phi' M'[0 \oplus' \phi^*(\xi^{(2)} \oplus x)].$$

But

$$\phi'(\eta^{(1)} \oplus' 0 \oplus 0) = \eta^{(1)} \oplus' 0 \oplus 0 \in X',$$

$$\phi'(0 \oplus' \tau^{(1)} \oplus 0) = 0 \oplus' \phi(\tau^{(1)} \oplus 0) \in X',$$

$$\phi' M'[0 \oplus' \phi^*(\xi^{(2)} \oplus x)] = 0 \oplus' V(\xi^{(2)} \oplus x) \in X'.$$

Thus,

$$(U' - V')x' = P_{-1}x' + I(U-V)Px' - \phi' Q_{-1}x'$$

and therefore

$$U' - V' = P_{-1} + I(U-V)P - \phi' Q_{-1}$$

as operators on X'. By what has been noted above, P_{-1} and $\phi' Q_{-1} \in \mathfrak{I}(X')$; since $U - V \in \mathfrak{I}(X)$ by hypothesis, it follows that $I(U-V)P \in \mathfrak{I}(X')$. We may therefore conclude that $U' - V' \in \mathfrak{I}(X')$. The result of Rosenblum-Kato-Kuroda mentioned previously now allows us to assert that U'_{ac} is unitarily equivalent to V'_{ac}. This completes the proof.

Corollary 1. *Let U and V be isometries on X with the Wold decompositions $U \cong S_\alpha \oplus U_u$ and $V \cong S_\alpha \oplus V_u$. If $U - V \in \mathfrak{I}(X)$ and α is a finite number, then U_{uac} and V_{uac} are unitarily equivalent.*

Proof. From the Theorem it follows that the dilations U', V' have unitarily equivalent absolutely continuous parts. *If B_α is the bilateral shift of multiplicity α, then*

$$U'_{ac} = (B_\alpha \oplus U_u)_{ac} \cong B_\alpha \oplus U_{uac}$$

and

$$V'_{ac} \cong (B_\alpha \oplus V_u)_{ac} \cong B_\alpha \oplus V_{uac}.$$

Since α is finite we deduce that $U_{uac} \cong V_{uac}$.

Does the conclusion of this corollary remain valid without assuming that α is finite? We do not know the answer to this question. However, we can prove the following result.

Proposition. *Let U and V be isometries on X such that $U - V$ has finite rank. Then $U_{u\text{ac}}$ and $V_{v\text{ac}}$ are unitarily equivalent.*

Proof. Let $U - V$ have the Schmidt series $\Sigma_{n=1}^{d} \phi_n \otimes \psi_n$ where $d =$ dimension $\mathcal{R}(U - V)$ is finite. Let $\mathfrak{M}$ denote the smallest (closed) subspace of X which is invariant for U and contains the vectors $\phi_1, \phi_2, \cdots, \phi_d$. Then $\mathfrak{M}$ is also invariant for V. Hence, $\mathfrak{M}^\perp$ is invariant for U^* and V^*. Moreover, the restriction operators on $\mathfrak{M}^\perp$ induced by U^* and V^* are equal. It then follows that their unitary parts, if any, are also equal. Hence, U and V have the same unitary parts on $\mathfrak{M}^\perp$. On the other hand, the operators U and V restricted to $\mathfrak{M}$ have index $\geq -d$ and so by Corollary 1 we may conclude that their unitary parts have the same absolutely continuous pieces. Now, by the Wold decomposition it is clear that a vector x belongs to the unitary part of an isometry I provided $\|I^{*n}x\| = \|x\|$ $(n = 1, 2, \cdots)$. Suppose y is such a vector for U which is orthogonal to the subspaces of $\mathfrak{M}$ and $\mathfrak{M}^\perp$ on which U is unitary. Set $y = y_{\mathfrak{M}} \oplus y_{\mathfrak{M}^\perp}$ with $y_{\mathfrak{M}} \in \mathfrak{M}$ and $y_{\mathfrak{M}^\perp} \in \mathfrak{M}^\perp$. By virtue of the Wold decomposition it follows that $\|U^{*n}y_{\mathfrak{M}}\| \to 0$ and $\|U^{*n}y_{\mathfrak{M}^\perp}\| \to 0$. Hence, $y = 0$. This concludes the proof.

Various authors [7], [8] have shown that if S is the simple unilateral shift and U is any unitary operator, then there exists a compact operator K such that $S + K$ is unitarily equivalent to $S \oplus U$. On the other hand the present author and J. Pincus [9] have recently shown that if U is any unitary operator with purely singular spectra, then there exists a trace class operator K such that $S + K$ is unitarily equivalent to $S \oplus U$.

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