THE ZYGMUND CONDITION FOR
POLYGONAL APPROXIMATION

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ABSTRACT. We investigate the class of functions which can be uniformly approximated to within $O(1/n)$ by a canonical choice of piecewise $(n-1)$-piece linear functions. The class turns out identical to the Zygmund class.

The problem of the rate of approximation by polygonal, or piecewise linear, functions was recently investigated in [1]. The gist of the story is that polygonal functions with vertices at $0$, $1/n$, $2/n$, $\ldots$, $1$ do roughly the same job as $n$th degree polynomials. For example $\text{Lip } \alpha (0 < \alpha < 1)$, is equivalent to approximability to within $n^{-\alpha}$.

What is missing is the case of $n^{-1}$. It was shown by Zygmund that for the polynomial case the condition for $n^{-1}$ accuracy is exactly that $f(x + h) - 2f(x) + f(x - h) = O(h)$. We show, in this note, that this very same condition is the correct one for polygonal fits.

Notation. $P_n(x)$ denotes any polygonal function with vertices at $0$, $1/n$, $2/n$, $\ldots$, $1$. Given any $f(x)$ we denote by $P_{n,f}(x)$ that particular polygonal function which agrees with it at $0$, $1/n$, $2/n$, $\ldots$, $1$. Finally we denote by the class $Z$ all continuous $f(x)$ such that, whenever $h > 0$, $x - h > 0$, $x + h < 1$, we have $|f(x + h) - 2f(x) + f(x - h)| \leq h$.

A. If $f(x) \in Z$ then $|f(x) - P_{n,f}(x)| \leq 1/2n$.

B. If, for every $n$, we can find a $P_n(x)$ for which $|f(x) - P_n(x)| \leq 1/48n$ throughout $[0, 1]$ then $f(x) \in Z$.

In the proofs we use the fact that if $x - h$, $x$, $x + h$ lie in an interval $[k/n, (k + 1)/n]$, then $P_n(x + h) - 2P_n(x) + P_n(x - h) = 0$ so that $|f(x + h) - 2f(x) + f(x - h)| \leq h$ is equivalent to $|r(x + h) - 2r(x) + r(x - h)| \leq h$ where $r = f - P_n$.

Proof of A. Let $x$ be a point where $|f - P_{n,f}|$ takes its maximum and set $k = [nx]$. Thus $x$ lies in $[k/n, (k + 1)/n]$ and we may assume without loss of

Received by the editors June 22, 1973.


Key words and phrases. Approximation, Zygmund class, piecewise linear functions.

1 Supported in part by AFOSR 72-2380A.

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generality that it lies in the first half of the interval. Choosing \( h = x - k/n \), then, insures that \( x - h, x, x + h \) all lie in \([k/n, (k + 1)/n]\). Thus we have, with \( r = f - P_n f \), that

\[
|r(x + h) - 2r(x) + r(x - h)| \leq h \leq 1/2n.
\]

Furthermore \( r(x - h) = r(k/n) = 0 \) while \( |r(x)| \geq |r(x + h)| \) so that \( |r(x)| \leq |r(x + h) - 2r(x) + r(x - h)| \leq 1/2n \) and the result follows upon remembering the maximality of \( r(x) \).

**Proof of B.** Let \( h > 0, x - h > 0, x + h < 1 \). Choose \( n \) as the largest integer for which \([n(x - h)] = [n(x + h)]\). It follows that \([2n(x - h)] < [2n(x + h)]\) and \([3n(x - h)] < [3n(x + h)]\) so that \((n(x - h), n(x + h))\) contains a fraction with denominator 2 and another fraction with denominator 3 (they are not the same fraction or else \((n(x - h), n(x + h))\) would contain an integer). The difference between these fractions being at least \( 1/6 \) gives the inequality \( 2nh \geq 1/6 \). Hence we have, with \( r(x) = f(x) - P_n(x) \),

\[
|r(x' + h) - 2r(x) + r(x - h)| \leq \frac{1}{48n} + \frac{2}{48n} + \frac{1}{48n} = \frac{1}{12n} \leq h.
\]

This suffices by our earlier remark since \( x - h, x, x + h \) all lie in \([k/n, (k + 1)/n]\) for \( k = [n(x - h)]\).

It is important to note that we really need all \( n \) in \( B \). Unlike the situation for polynomials, approximation can hold for all even \( n \) without \( Z \). For \( f(x) = |x - \frac{1}{2}| \log(1/|x - \frac{1}{2}|) \) we get \( |f - P_{n,f}| \leq c/n \) for \( n \) even, but

\[
f\left(\frac{1}{2} + h\right) - 2f\left(\frac{1}{2}\right) + f\left(\frac{1}{2} - h\right) = 2h \log(1/h) \neq O(h).
\]

**REFERENCE**


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