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THE ZYGMUND CONDITION FOR POLYGONAL APPROXIMATION

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ABSTRACT. We investigate the class of functions which can be uniformly approximated to within $O(1/n)$ by a canonical choice of piecewise (n-piece) linear functions. The class turns out identical to the Zygmund class.

The problem of the rate of approximation by polygonal, or piecewise linear, functions was recently investigated in [1]. The gist of the story is that polygonal functions with vertices at $0, 1/n, 2/n, \cdots, 1$ do roughly the same job as nth degree polynomials. For example Lip α $(0 < \alpha < 1)$, is equivalent to approximability to within $n^{-\alpha}$.

What is missing is the case of n^{-1}. It was shown by Zygmund that for the polynomial case the condition for n^{-1} accuracy is exactly that $f(x+h) - 2f(x) + f(x-h) = O(h)$. We show, in this note, that this very same condition is the correct one for polygonal fits.

Notation. $P_n(x)$ denotes any polygonal function with vertices at $0, 1/n, 2/n, \cdots, 1$. Given any $f(x)$ we denote by $P_{n,f}(x)$ that particular polygonal function which agrees with it at $0, 1/n, 2/n, \cdots, 1$. Finally we denote by the class Z all continuous $f(x)$ such that, whenever $h > 0$, $x - h > 0$, $x + h < 1$, we have $|f(x+h) - 2f(x) + f(x-h)| \le h$.

A. *If* $f(x) \in Z$ *then* $|f(x) - P_{n,f}(x)| \le 1/2n$.

B. *If, for every* n, *we can find a* $P_n(x)$ *for which* $|f(x) - P_n(x)| \le 1/48n$ *throughout* $[0, 1]$ *then* $f(x) \in Z$.

In the proofs we use the fact that if $x - h$, x, $x + h$ lie in an interval $[k/n, (k+1)/n]$, then $P_n(x+h) - 2P_n(x) + P_n(x-h) = 0$ so that $|f(x+h) - 2f(x) + f(x-h)| \le h$ is equivalent to $|r(x+h) - 2r(x) + r(x-h)| \le h$ where $r = f - P_n$.

Proof of A. Let x be a point where $|f - P_{n,f}|$ takes its maximum and set $k = [nx]$. Thus x lies in $[k/n, (k+1)/n]$ and we may assume without loss of

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generality that it lies in the first half of the interval. Choosing $h = x - k/n$, then, insures that $x - h$, x, $x + h$ all lie in $[k/n, (k+1)/n]$. Thus we have, with $r = f - P_{n,f}$, that $|r(x+h) - 2r(x) + r(x-h)| \le h \le 1/2n$. Furthermore $r(x-h) = r(k/n) = 0$ while $|r(x)| \ge |r(x+h)|$ so that $|r(x)| \le |r(x+h) - 2r(x) + r(x-h)| \le 1/2n$ and the result follows upon remembering the maximality of $r(x)$.

Proof of B. Let $h > 0$, $x - h > 0$, $x + h < 1$. Choose n as the largest integer for which $[n(x-h)] = [n(x+h)]$. It follows that $[2n(x-h)] < [2n(x+h)]$ and $[3n(x-h)] < [3n(x+h)]$ so that $(n(x-h), n(x+h))$ contains a fraction with denominator 2 and *another* fraction with denominator 3 (they are not the same fraction or else $(n(x-h), n(x+h))$ would contain an integer). The difference between these fractions being at least $1/6$ gives the inequality $2nh \ge 1/6$. Hence we have, with $r(x) = f(x) - P_n(x)$,

$$|r(x+h) - 2r(x) + r(x-h)| \le \frac{1}{48n} + \frac{2}{48n} + \frac{1}{48n} = \frac{1}{12n} \le h.$$

This suffices by our earlier remark since $x - h$, x, $x + h$ all lie in $[k/n, (k+1)/n]$ for $k = [n(x-h)]$.

It is important to note that we really need all n in B. Unlike the situation for polynomials, approximation can hold for all even n without Z. For $f(x) = |x - \frac{1}{2}| \log(1/|x - \frac{1}{2}|)$ we get $|f - P_{n,f}| \le c/n$ for n even, but

$$f(\tfrac{1}{2} + h) - 2f(\tfrac{1}{2}) + f(\tfrac{1}{2} - h) = 2h \log(1/h) \ne O(h).$$

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