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NOTE ON A VOLTERRA INTEGRO-DIFFERENTIAL EQUATION SYSTEM

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ABSTRACT. This note concerns a type of nonlinear Volterra integro-differential equation system which has only asymptotically zero solutions. It is a simple generalization of a one-dimensional result of J. J. Levin. It is a correction to another such note and gives a counterexample to that note.

In [1] Kemp discusses the integro-differential equation

$$x'(t) = -\int_0^t a(t-w)g(x(w))\,dw \tag{1}$$

where x maps $[0, \infty)$ into R^n, g maps R^n into R^n and a maps $[0, \infty)$ into n by n matrices over R. Take the following as basic assumptions:

(a) $g \in C(R^n)$, $x^T g(x) > 0$ for $x \neq 0$ (where x^T is the transpose of the n by 1 matrix x), there exists a scalar function $G \in C'(R^n)$ such that g is the gradient of G and $G(x) \to \infty$ as $|x| \to \infty$;

(b) $a \in C[0, \infty)$, and $(-1)^k a^{(k)}(t)$ is a real symmetric positive semidefinite matrix for $0 < t < \infty$, $k = 0, 1, 2, 3$.

Under (a) and (b), Kemp states the following theorem about asymptotic behavior.

Theorem I. *Any solution $u(t)$ of (1) satisfies*

$$\lim_{t \to \infty} u^{(j)}(t) = 0 \qquad (j = 0, 1, 2) \tag{2}$$

provided that $a(t) \not\equiv a(0)$.

However the system below is a counterexample to that theorem.

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$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = -\int_0^t \begin{bmatrix} \exp(-(t-w)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(w) \\ u_2(w) \end{bmatrix} dw, \quad u_1(0) = 0; \quad u_2(0) = 1.$$

Kemp's result is an attempt at generalizing a one-dimensional theorem of J. J. Levin which is in [2]. In the one-dimensional argument the following result is important (Lemma 2 in [2]).

Lemma II. *If* (b) *is satisfied and if* $a(t) \not\equiv a(0)$ *then either* $-a'(t)$, $a''(t) > 0$ *for* $0 < t < \infty$ *or there exists a* $t_0 > 0$ *such that* $-a'(t), a''(t) > 0$ *for* $0 < t < t_0$ *and* $a(t) \equiv a(t_0) = a(\infty) > 0$ *for* $t_0 \leq t < \infty$.

Levin uses this lemma to show that there is an $S > 0$ such that $a''(t) > 0$ for $0 < t \leq S$. The lemma, however, does not hold for the n-dimensional case. To generalize Levin's theorem we may use Lemma III.

Lemma III. *If* (b) *is satisfied and if there is no nonzero* $x \in R^n$ *such that* $x^T a(t)x = x^T a(0)x$ *for all* t, $0 \leq t < \infty$, *then there exists an* $S > 0$ *such that* $a''(t)$ *is positive definite for* $0 < t \leq S$.

With Lemma III all of Levin's arguments may be put through, with a little extra work, to give Theorem IV.

Theorem IV. *If* (a) *and* (b) *hold and* $u(t)$ *is any solution of* (1), *then* $u(t)$ *satisfies* (2) *provided that there is no nonzero* $x \in R^n$ *such that* $x^T a(t)x = x^T a(0)x$ *for all* t, $0 \leq t < \infty$.

Proof of Lemma III. The function $x^T a(t)x$, $x \neq 0$, satisfies the lemma of Levin. Therefore for any $x \in R^n$, $x \neq 0$, there exists $t(x)$ such that $x^T a''(t)x > 0$ for $0 < t < t(x)$. Note that $a''(t)$ is positive definite if and only if for all $x \in R^n$, $|x| = 1$, $x^T a''(t)x > 0$.

For a contradiction proof assume that there exist a sequence x_n, $|x_n| = 1$, and a sequence t_n, $t_n > 0$ and $\lim t_n = 0$, such that $x_n^T a''(t_n)x_n = 0$. Without loss of generality assume $\lim x_n = y$ with $|y| = 1$. Choose T such that $0 < T < t(y)$ so that $y^T a''(T)y > 0$. $\operatorname{Lim} x_n^T a''(T)x_n = y^T a''(T)y > 0$ implies there is some N such that for $n > N$, $x_n^T a''(T)x_n > 0$. $x_n^T a''(t)x_n$ is nonincreasing in t for $t > 0$ and any n (just look at its derivative); hence $x_n^T a''(T)x_n > 0$ for $n > N$ implies $x_n^T a''(t)x_n > 0$ for $0 < t < T$ and $n > N$. Now if n is sufficiently big such that $n > N$ and $0 < t_n < T$, we have two contradictory statements: $x_n^T a''(t_n) x_n = 0$ and $x_n^T a''(t_n) x_n > 0$. Thus there is a contradiction and there must exist an S as described.

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