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ON DIRECT SUMS OF REDUCTIVE OPERATORS

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ABSTRACT. An example is given to show that the direct sum of two (distinct) reductive operators need not be reductive. The conjecture that $A \oplus A$ is reductive if A is reductive, is shown to be equivalent to the reductive operator conjecture (every reductive operator is normal).

Let A be a bounded linear operator (briefly, an operator) on a Hilbert space H. The operator A is said to be reductive if every invariant subspace for A reduces A.

The theorem stated below can be derived from a result of Radjavi and Rosenthal (see [2, Lemma 2]). In this paper, a new and simple proof will be given.

Theorem. *The following statements are equivalent*:

(a) *If A is reductive, then A is normal.*

(b) *If A is reductive, then $A \oplus A$ is reductive* (*on* $H \oplus H$).

To prove that (a) implies (b), we make use of the following result of Sarason [4].

Lemma. *Let A be a normal operator. Then A is reductive if and only if the weakly closed algebra with identity generated by A, denoted by $W(A)$, is a star-algebra.*

It is a straightforward argument to show that $W(A \oplus A)$ is a star-algebra if $W(A)$ is a star-algebra. Hence (a) implies (b).

To prove the converse, we let $M = \{x \oplus Ax\colon x \text{ is in } H\}$, the graph of A. The subspace M is invariant for $A \oplus A$. If $A \oplus A$ is reductive, then $(A^* \oplus A^*)(x \oplus Ax) = A^*x \oplus A^*Ax$ is in M for each x in H. Thus $A^*Ax = AA^*x$ and A is normal. This shows that (b) implies (a).

Although it is not known whether or not $A \oplus A$ is reductive whenever A is reductive, the following example shows that a more general statement is false.

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Example. A bilateral shift U is a direct sum of two reductive operators, hence a direct sum of two (distinct) reductive operators need not be reductive.

Use the spectral theorem for normal operators to split U into a direct sum $U_1 \oplus U_2$ in such a manner that 0 is in the unbounded component of each of the resolvent sets $\rho(U_1)$ and $\rho(U_2)$. Then U_1 and U_2 are easily seen to be reductive (see Sarason [3, Theorem 11]), while $U = U_1 \oplus U_2$ is not reductive.

Dyer, Pedersen, and Porcelli [1] have shown that statement (a) of the Theorem above is logically equivalent to the invariant subspace conjecture. Thus, statement (b) of the Theorem gives another equivalence to the invariant subspace conjecture.

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