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SOME FORMULAS IN THE STEENROD ALGEBRA

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ABSTRACT. In this paper we prove that certain sums of Steenrod operations are zero. We also show how certain Milnor basis elements can be expanded in the Adem basis.

1. **Introduction.** The canonical anti-automorphism χ in the Steenrod algebra is defined by the equation $\Sigma_{i=0}^{k} \chi(\mathrm{Sq}^i)\mathrm{Sq}^{k-i} = 0$ if $k > 0$. For reasons unrelated to this paper, we wanted to know the value of $\Sigma_{i=0}^{k} \mathrm{Sq}^i\mathrm{Sq}^{k-i}$. Our study led to the results of this paper. Special cases of our main results (Theorems 3.1 and 4.1) show that

$$\sum_{i=0}^{k} \mathrm{Sq}^i\mathrm{Sq}^{k-i} = 0 \quad \text{if } k \not\equiv 0(3)$$

and, if $k \equiv 0(3)$,

$$\sum_{i=0}^{k} \mathrm{Sq}^i\mathrm{Sq}^{k-i} = \mathrm{Sq}^{(0,k/3)} = \sum_{i=2k/3}^{k} \mathrm{Sq}^i\mathrm{Sq}^{k-i}.$$

2. $\mathrm{Hom}_k(A, A)$. Let k be a field and let A be a graded, connected, Hopf algebra over k. Let $H = \mathrm{Hom}_k(A, A)$ be all vector space maps f of degree 0 such that $f_0: A_0 \to A_0$ is $\eta_0\epsilon_0$. If $f, g \in H$, define $f * g \in H$ by $f * g = \phi(f \otimes g)\psi$. Let $0 \in H$ be the map which is 0 in positive dimensions and $\eta_0\epsilon_0$ in dimension 0. The following propositions follow easily.

Proposition 2.1. $0 * f = f * 0 = f$.

Proposition 2.2. *If A is commutative or cocommutative, $f * g = g * f$.*

Proposition 2.3. *If A is associative and coassociative, then $*$ is associative.*

Proposition 2.4. *If f is an algebra map, then $f(g * h) = (fg) * (fh)$. If f is a coalgebra map, then $(g * h)f = (gf) * (hf)$.*

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Proposition 2.5. *If f and g are algebra maps and A is commutative, then $f * g$ is an algebra map. If f and g are coalgebra maps and A is cocommutative, then $f * g$ is a coalgebra map.*

Proposition 2.6. *Under the assumptions of Propositions 2.2 and 2.3,* $\mathrm{Hom}_k(A, A)$ *is an abelian group under* $*$.

3. **The case A is the Steenrod algebra.** As an example of some of these notions, we look at the case when $A = \mathcal{A}_p$, the mod p Steenrod algebra. Let $\mathrm{id} \in H$ denote the identity map, and define $t_r \in H$ by $t_1 = \mathrm{id}$ and $t_r = t_{r-1} * \mathrm{id}$. For example, $t_2 = \mathrm{id} * \mathrm{id}$ and

$$t_2(\mathrm{Sq}^k) = \sum_{i=0}^{k} \mathrm{Sq}^i \mathrm{Sq}^{k-i} \quad \text{if } p = 2.$$

Let Sq^R and $\mathcal{P}^R$ be the Milnor basis elements [2] corresponding to sequences $R = (r_1, \cdots)$. Let $\mathrm{Sq}_r(k)$ and $\mathcal{P}_r(k)$ be the elements corresponding to the sequence $(0, \cdots, 0, k, 0, \cdots)$, with the nonzero term in the rth place. Let $p_k = 1 + p + \cdots + p^{k-1}$. Let $\mathcal{P} = 1 + \mathcal{P}^1 + \mathcal{P}^2 + \cdots$. If $S = (s_1, \cdots)$ is a Milnor basis sequence, define $\binom{r}{S} = \binom{r}{1}^{s_1}\binom{r}{2}^{s_2}\cdots$, with the convention that $(0)^0 = 1$.

Theorem 3.1. $t_r(\mathcal{P}) = \Sigma_S \binom{r}{S} \mathcal{P}^S$.

Corollary 3.2. *If* $p = 2$, $t_2(\mathrm{Sq}) = \Sigma_{k=0}^{\infty} \mathrm{Sq}_2(k)$.

In order to prove Theorem 3.1, we study $t_r^*\colon \mathcal{A}^* \to \mathcal{A}^*$, the dual map. This is a map of algebras by Proposition 2.5. Let $I \subset \mathcal{A}^*$ be the ideal generated by $\xi_2, \xi_3, \cdots, \tau_0, \tau_1, \cdots$. We first prove the following lemma.

Lemma 3.3. $t_r^*(\xi_k) \equiv \binom{r}{k}(\xi_1)^{p_k} \bmod I$ *if* $k > 0$. $t_r^*(\tau_k) \equiv 0 \bmod I$ *if* $k \geq 0$.

Proof. We prove this by induction on r. t_1^* is the identity and the result is true. In general,

$$t_r^*(\xi_k) = \sum_{i=0}^{k} t_{r-1}^*(\xi_{k-i})^{p^i} \xi_i \equiv t_{r-1}^*(\xi_k)\xi_0 + t_{r-1}^*(\xi_{k-1})^p \xi_1 \bmod I$$

$$\equiv \binom{r-1}{k}(\xi_1)^{p_k} + \binom{r-1}{k-1}^p (\xi_1)^{p \cdot p_{k-1}+1} \bmod I$$

$$= \binom{r}{k}(\xi_1)^{p_k} \bmod I.$$

Also, $t_r^*(\tau_k) \equiv 0 \bmod I$ as it is odd dimensional.

Proof of 3.1. Let $m = \xi_1^{s_1} \cdots \xi_k^{s_k} \in \mathcal{A}^*$. Then $\langle t_r(\mathcal{P}), m\rangle = \langle \mathcal{P}, t_r^*(m)\rangle = \binom{r}{1}^{s_1} \cdots \binom{r}{k}^{s_k}$ by Lemma 3.3. If m involves one of the τ's, $\langle t_r(\mathcal{P}), m\rangle = 0$. This proves Theorem 3.1.

4. $\mathcal{P}^{(0,k)}$. Motivated by Corollary 3.2 and the fact that $\mathcal{P}_r(k)$ plays an important role in the study of modules over the Steenrod algebra (see [1] and [3]), we will show how $\mathcal{P}^{(0,k)}$ can be expanded in the Adem basis for $\mathcal{A}_p$.

Theorem 4.1. $\mathcal{P}^{(0,k)} = \sum_{s=0}^{k} (-1)^{s+k}\mathcal{P}^{(p+1)k-s}\mathcal{P}^{s}$ = *the signed sum of all admissible Adem basis elements of length* ≤ 2 *and of degree* $k(p+1)(2p-2)$.

Before proving Theorem 4.1, we need a couple of lemmas and a corollary.

Lemma 4.2. $\sum_{t=0}^{m}(-1)^{t+1}\binom{m+f}{t} = (-1)^{m+1}\binom{m+f-1}{m}$ *if* $f > 0$.

Lemma 4.3. $\binom{(p+1)m-1}{m} \equiv 0(p)$.

Corollary 4.4. $\sum_{t=0}^{m}(-1)^{t+1}\binom{(p+1)m}{t} \equiv 0(p)$.

Proof of 4.2. $\sum_{t=0}^{m}(-1)^{t+1}\binom{m+f}{t} = \sum_{t=0}^{m}(-1)^{t+1}\left[\binom{m+f-1}{t} + \binom{m+f-1}{t-1}\right] = (-1)^{m+1}\binom{m+f-1}{m}$ by telescoping.

Proof of 4.3. Let $m = b_0p^a + b_1p^{a+1} + \cdots$, $0 < b_0 < p$, be the p-adic expansion of m. Then $(p+1)m = b_0p^a + (b_1+1)p^{a+1} + \cdots$ and $(p+1)m-1 = (p-1) + (p-1)p + \cdots + (p-1)p^{a-1} + (b_0-1)p^a + (b_1+1)p^{a+1} + \cdots$ and hence $\binom{(p+1)m-1}{m} \equiv 0(p)$ as the term corresponding to p^a is

$$\binom{b_0-1}{b_0} \equiv 0(p).$$

Proof of 4.4. Corollary 4.4 follows from Lemmas 4.2 and 4.3 by taking $f = pm$.

We now prove Theorem 4.1. We use Milnor's theorem [2] that

$$\mathcal{P}^t\mathcal{P}^s = \sum_{i=0}^{s} \binom{t+s-(p+1)i}{s-i}\mathcal{P}^{(t+s-(p+1)i,\,i)}.$$

Hence,

$$\sum_{s=0}^{k} (-1)^s \mathcal{P}^{(p+1)k-s}\mathcal{P}^s = \sum_{s=0}^{k}\sum_{i=0}^{s} (-1)^s \binom{(p+1)(k-i)}{s-i} \mathcal{P}^{((p+1)(k-i),i)}$$

$$= \sum_{i=0}^{k-1}\left(\sum_{s=i}^{k} (-1)^s \binom{(p+1)(k-i)}{s-i}\right) \mathcal{P}^{((p+1)(k-i),i)} + (-1)^k \mathcal{P}^{(0,k)}.$$

But

$$\sum_{s=i}^{k} (-1)^s \binom{(p+1)(k-i)}{s-i} = (-1)^{i-1} \sum_{t=0}^{m} (-1)^{t+1} \binom{(p+1)m}{t} \equiv 0(p)$$

by Corollary 4.4. Hence the theorem is proved.

Corollary 4.5. *If* $k \not\equiv 0(3)$, *then* $\sum_{i=0}^{k} \mathrm{Sq}^i \mathrm{Sq}^{k-i} = 0$. *If* $k \equiv 0(3)$, *then* $\sum_{i=0}^{2k/3-1} \mathrm{Sq}^i \mathrm{Sq}^{k-i} = 0$.

Proof. The first part follows directly from Corollary 3.2 and the second part follows from Corollary 3.2 and Theorem 4.1.

Theorem 4.1 might lead one to conjecture that $\mathcal{P}_r(k)$ = the signed sum of all admissible Adem basis elements of length $\leq r$ and of degree $p_r k$. Calculation shows that this is true for $(k = 1, r = 3, p = 2)$, but false for $(k = 2 \text{ or } 3, r = 3, p = 2)$, $(k = 1, r = 3, p = 3)$, and $(k = 1, r = 4, p = 2)$.

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