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STEINHAUS TYPE THEOREMS FOR SUMMABILITY MATRICES

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ABSTRACT. Necessary and sufficient conditions are given for an infinite matrix to sum all bounded strongly summable sequences. It is shown that the Borel matrix does not sum all such sequences. A corollary is that the bounded summability field of the Borel method is strictly contained in that of the $(C, 1)$ method. Also, it is proved that no coregular matrix can almost sum all bounded sequences—a generalization of Steinhaus' theorem.

1. **Introduction.** The well-known theorem of Steinhaus asserts that no Toeplitz matrix can sum all bounded sequences. A similar type of result due to Kuttner [4] states that, for $0 < p < 1$, no Toeplitz matrix can sum all sequences in w_p (the space of all strongly $(C, 1)$ summable sequences with index p). These theorems of Steinhaus and Kuttner are also valid if Toeplitz matrices are replaced by coregular matrices [6].

It may be observed that the sequence in [4], which is in w_p but is not summable by the Toeplitz matrix, is unbounded. Consequently it is natural to consider whether or not a Toeplitz matrix can sum all bounded sequences in w_p. It is easy to see that there are Toeplitz matrices which do sum all bounded sequences in w_p. In fact if $p > 0$ and $r > 0$ then it is well known that, for bounded sequences, w_p is equal to w_1 and (C, r) is equivalent to $(C, 1)$. Hence, if l_∞ denotes the space of bounded sequences, then $l_\infty \cap w_p \subset l_\infty \cap (C, r)$, so that the Toeplitz matrix of (C, r) means sum every sequence in $l_\infty \cap w_p$. Obviously the unit matrix does not sum every sequence in $l_\infty \cap w_p$, and it is easy to see that there are Toeplitz matrices not equivalent to convergence which do not sum every sequence in $l_\infty \cap w_p$. These observations lead us to characterize (in §3) all infinite matrices which sum every sequence in $l_\infty \cap w_p$.

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In §4 we consider Steinhaus type theorems in which summability by a matrix is replaced by 'almost summability'.

2. **Definitions.** By l_∞ we denote the Banach space of all bounded complex sequences $x = (x_k)$ with $\|x\| = \sup |x_k|$. By c and f we denote the closed subspaces of convergent and almost convergent sequences. The almost convergent sequences f were introduced by Lorentz [5] via Banach limits. With strict inclusion we have $c \subset f \subset l_\infty$.

For $p > 0$ we define w_p to be the space of all x such that, for some number l,

$$\frac{1}{n}\sum_{k=1}^{n} |x_k - l|^p \to 0 \qquad (n \to \infty). \tag{1}$$

The space w_p has been considered by several authors [1], [4], [6].

If $A = (a_{nk})$ is an infinite matrix of complex numbers we write $A_n(x) = \Sigma a_{nk}x_k$, all sums being over $1 \leq k \leq \infty$, unless otherwise indicated. We write $Ax = (A_n(x))$.

For nonempty sets of sequences X, Y we denote by (X, Y) the set of all A such that $Ax \in Y$ for every $x \in X$. Also, we define $X(A) = \{x\colon Ax \in X\}$, so for example $c(A)$ is the summability field of A, and $f(A)$ is the set of all x which are almost summable by A. Some results on almost summability may be found in [3].

It is familiar that $A \in (c, c)$ if and only if $\|A\| = \sup_n \Sigma|a_{nk}| < \infty$, $\lim_n a_{nk} = a_k$ (each k), $\lim_n \Sigma a_{nk} = a$. The space (c, c) of all conservative matrices is a Banach algebra under the matrix product, and the characteristic χ defined by $\chi(A) = a - \Sigma a_k$ is a scalar homomorphism on (c, c). The kernel of χ is written K_0 and is called the conull matrices. The set $(c, c) - K_0$ of coregular matrices is written as K.

Finally, a set E of positive integers is said to be of zero density if the number of elements of E which lie in $[1, n]$ is $o(n)$ as $n \to \infty$. For example, $\{2, 2^2, 2^3, \cdots\}$ is of zero density.

3. **Summability of bounded strongly summable sequences.**

Theorem 1. *Let $p > 0$. Then $A \in (l_\infty \cap w_p, c)$ if and only if A is conservative and*

$$\sum_{k \in E} |a_{nk} - a_k| \to 0 \qquad (n \to \infty) \tag{2}$$

for each set E of zero density.

Proof. For the sufficiency let $x \in l_\infty$ and suppose (1) holds. Since A is conservative it is sufficient to show that

$$\sum |a_{nk} - a_k| \cdot |x_k - l| \to 0 \qquad (n \to \infty). \tag{3}$$

Let $\epsilon > 0$ and write $E = \{k: |x_k - l| \geq \epsilon\}$. Since $x_k \to l(w_1)$ it is clear that E has zero density. Hence, for all n,

$$\sum_{k\in E} |a_{nk} - a_k| \cdot |x_k - l| + \sum_{k\in -E} |a_{nk} - a_k| \cdot |x_k - l|$$

$$\leq 2\|x\| \cdot \sum_{k\in E} |a_{nk} - a_k| + 2\epsilon \|A\|.$$

Now letting $n \to \infty$ we see that (3) holds.

Consider now the necessity. From $A \in (l_\infty \cap w_p, c)$ and the fact that $c \subset l_\infty \cap w_p$ it follows that $A \in (c, c)$. Suppose, if possible, that there exists a set E of zero density such that (2) fails. This implies that E is an infinite set, so we write $E = \{e_1, e_2, \cdots\}$. By the proof of Schur's theorem [7, p. 169] there is a bounded sequence $z = (z_{e_1}, z_{e_2}, \cdots)$ such that the sequence $(\Sigma_{k\in E} (a_{nk} - a_k) z_k)$ is divergent. Now define a sequence x by $x_k = z_k$ for $k = e_i$ and $x_k = 0$ for $k \neq e_i$. Then $x \in l_\infty$, and since E has zero density we see that $x_k \to 0\ (w_p)$. Hence there exists $x \in l_\infty \cap w_p$ such that $x \notin c(A)$, which is contrary to $A \in (l_\infty \cap w_p, c)$. This proves the theorem.

As an application of Theorem 1 we consider the Borel matrix defined by $a_{nk} = e^{-n} n^k / k!$, with $n, k \geq 0$.

Theorem 2. *The Borel matrix does not sum all sequences in* $l_\infty \cap w_p$.

Proof. Define $E = \{e_1, e_2, \cdots\}$, where $e_k = 4^i + k - 2^i$ for $2^i \leq k < 2^{i+1}$ and $i = 0, 1, 2, \cdots$. Thus $E = \{1, 4, 5, 16, 17, 18, 19, 64, 65, \cdots\}$. Now take $n \geq 1$ and determine i such that $4^i \leq n < 4^{i+1}$. Then the number of elements of E which lie in $[1, n]$ is less than or equal to $1 + 2 + \cdots + 2^i < 2n^{1/2} = o(n)$, so that E is of zero density. Now let (a_{nk}) be the Borel matrix, and write $m = 4^n$. Then

$$\sum_{k\in E} |a_{mk}| > \sum_{h=0}^{m^{1/2}-1} a_{m,m+h}. \tag{4}$$

Using the simple inequality $p! \leq e^{-p+1} p^p p^{1/2}$, valid for positive integers p, we find that

$$\begin{aligned}\log a_{m,m+h} &\geq -m + (m+h)\log m - (m+h)\log(m+h) \\ &\quad - \log(m+h)^{1/2} + m + h - 1 \\ &= -(m+h)\log(1 + h/m) - \log(m+h)^{1/2} + h - 1 \\ &\geq -h - h^2/m - \log(2m)^{1/2} + h - 1,\end{aligned}$$

whence

$$a_{m,m+h} \geq \exp(-h^2/m)/e(2m)^{1/2} \quad \text{for } 0 \leq h < m^{1/2}.$$

By (4) it now follows that

$$\sum_{k \in E} |a_{mk}| > \frac{1}{e(2m)^{1/2}} \int_0^{m^{1/2}} e^{-t^2/m}\, dt = \frac{1}{e8^{1/2}} \int_0^1 e^{-u} u^{-1/2}\, du,$$

which implies that $\limsup_n \Sigma_{k \in E} |a_{nk}| > 0$. By Theorem 1 it follows that $A \notin (l_\infty \cap w_p, c)$.

An immediate consequence of Theorem 2 is that if S is any summability method which is included in the Borel method then $S \notin (l_\infty \cap w_p, c)$. In particular this applies when $S = (E, q)$, the Euler method [2, Theorem 128]. Another corollary to Theorem 2 is that the bounded summability field of the Borel method is strictly contained in the bounded summability field of the $(C, 1)$ method. To see this we write $A_n = a_0 + a_1 + \cdots + a_n$ and assume that (A_n) is bounded and Borel summable. Hence $a_n = o(n)$, so by [2, Theorem 147], with $\rho = 1$, we have that (A_n) is $(C, 3)$ summable, and so $(C, 1)$ summable since (A_n) is bounded. Thus we have $l_\infty \cap \text{Borel} \subset l_\infty \cap (C, 1)$. If equality occurred in this last inclusion then we would obtain a contradiction by Theorem 2.

4. **Almost summability of bounded sequences.** Steinhaus' theorem may be put in the following form: if A is coregular then the inclusion $l_\infty \subset c(A)$ is false. Now for any matrix A we have $c(A) \subset f(A)$, so that it seems possible that the inclusion $l_\infty \subset f(A)$ might be true for coregular matrices. However, in Theorem 3 (iii) we show otherwise. Consequently, we have a generalization of Steinhaus' theorem, viz. that no coregular matrix can almost sum all bounded sequences.

Theorem 3. *Let A and B be conservative matrices and suppose that $A \in (l_\infty, c(B))$. Then*

(i) $BA \in K_0$.

(ii) *If $B \in K$ then $A \in K_0$.*

(iii) *If $A \in K$ then $l_\infty \subset f(A)$ is false.*

Proof. (i) Since $A \in (l_\infty, c(B))$ we have $B(Ax) \in c$ for all $x \in l_\infty$. Now A and B conservative implies $B(Ax) = (BA)x$ for all $x \in l_\infty$, whence $(BA)x \in c$ for all $x \in l_\infty$, so that $BA \in (l_\infty, c) \subset K_0$.

(ii) By (i) and the fact that χ is a scalar homomorphism we have $\chi(B)\chi(A) = 0$, whence the result.

(iii) If $A \in K$ but $l_\infty \subset f(A)$, then $A \in (l_\infty, f)$ which implies $A \in (l_\infty, (C, 1))$. Hence, by (ii), on taking B to be the coregular $(C, 1)$ matrix we obtain $A \in K_0$, contrary to $A \in K$.

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