PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

AN EXTENSION OF THE HAUSDORFF-YOUNG THEOREM

ROBERT M. YOUNG

ABSTRACT. Using the Riesz-Thorin interpolation theorem, we show that if $1 < p < 2$ and f belongs to $L^p(-\pi, \pi)$, then $\{\hat{f}(z_n)\}$ belongs to l^q ($q = p/(p-1)$) for a very general class of complex sequences $\{z_n\}$. We also obtain a convergence criterion for a related class of exponential sums.

1. **Introduction.** The classical Hausdorff-Young theorem states that if $1 < p < 2$ and f belongs to $L^p(-\pi, \pi)$, then $\{\hat{f}(n)\}$ belongs to l^q, where q is the conjugate exponent, that is $q = p/(p-1)$. In this note we offer a simple proof showing that $\{\hat{f}(z_n)\}$ belongs to l^q for a very general class of complex sequences $\{z_n\}$. We also obtain a convergence criterion for a related class of exponential sums. Our results, although seemingly known, do not appear to be in the literature.

Definition. A sequence $\{z_n\}$ of distinct complex numbers will be called *separated* if there is a constant $\delta > 0$ such that $|z_n - z_m| \geq \delta$ for all $n \neq m$.

Theorem. *Let $\{z_n\}$ be a separated sequence of points lying in a strip parallel to the real axis. Let $1 < p < 2$ and let q be the conjugate exponent.*

(i) *There is a constant A such that the inequality*

$$\left\| \sum c_n e^{iz_n t} \right\|_q \leq A \left(\sum |c_n|^p \right)^{1/p} \tag{1}$$

holds whenever $\{c_n\}$ belongs to l^p.

(ii) *If $f \in L^p(-\pi, \pi)$, then*

$$\left(\sum |\hat{f}(z_n)|^q \right)^{1/q} \leq A \|f\|_p, \tag{2}$$

with A as above.

2. **Preliminary lemma.** The following lemma was proved by Titchmarsh [4] for the case when z_n is real and later reproved by Paley and Weiner [3] and Ingham [1]. The proof in the general case is a simple extension of the argument given in [1] and is therefore omitted.

Received by the editors September 1, 1972.

AMS (MOS) subject classifications (1970). Primary 42A16; Secondary 40A05.

Key words and phrases. Exponential sums, separated sequence, almost periodic functions, interpolation of operators.

Lemma. *Let $\{z_n\}$ be a separated sequence of points lying in a strip parallel to the real axis. There is a constant A such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum c_n e^{iz_n t}\right|^2 dt \le A\sum |c_n|^2 \tag{3}$$

whenever $\Sigma|c_n|^2 < \infty$.

3. **Proof of the theorem.** If $\{c_n\} \in l^1$, then

$$\left|\sum c_n e^{iz_n t}\right| \le A\sum |c_n| \tag{4}$$

for some absolute constant A. The inequalities (3), (4) show that the mapping $T: l^2 \to L^2$ given by $\{c_n\} \mapsto \Sigma c_n e^{iz_n t}$ is continuous and that the restriction mapping $T: l^1 \to L^\infty$ is also continuous. It follows from the Riesz-Thorin theorem [2, p. 97] that the restriction of T to l^p is a bounded operator from l^p into L^q, and this establishes (1).

Now inequality (2) follows immediately from (1) since we can choose complex numbers d_n, with $\Sigma|d_n|^p = 1$, so that

$$\left(\sum |\hat{f}(z_n)|^q\right)^{1/q} = \sum \hat{f}(z_n)d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\sum d_n e^{iz_n t}\, dt$$

$$\le \|f\|_p \left\|\sum d_n e^{iz_n t}\right\|_q \le A\|f\|_p.$$

4. **Remark.** When $\Sigma|c_n|^2 < \infty$ the series $\Sigma c_n e^{iz_n t}$ converges in mean square over every interval $(x, x+1)$, uniformly with respect to x, and hence represents a function which is almost periodic in the sense of Wiener and Stepanoff. (See [3] and [4].)

Acknowledgement. The author wishes to thank the referee for his simplification of part (ii) of the Theorem.

BIBLIOGRAPHY

1. A. E. Ingham, Some *trigonometrical inequalities with applications to the theory of series*, Math. Z. **41** (1936), 367–379.

2. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR **40** #1734.

3. R. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R. I., 1934.

4. E. C. Titchmarsh, *A class of trigonometrical series*, J. London Math. Soc. 3 (1928), 300–304.

DEPARTMENT OF MATHEMATICS, OBERLIN COLLEGE, OBERLIN, OHIO 44074