PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August, 1974

ZERO-ONE LAWS FOR STABLE MEASURES

R. M. DUDLEY[1] AND MAREK KANTER

ABSTRACT. For any stable measure μ on a vector space, every measurable linear subspace has measure 0 or 1.

1. **Introduction.** It is known that for any Gaussian probability measure, a linear subspace has measure 0 or 1. This result has been extended to additive subgroups by Kallianpur [2]. Here we extend the zero-one law in a different direction, replacing "Gaussian" by "stable". We begin with some definitions.

Definition. Let S be a vector space over $\mathbf{R}$ and let $\mathcal{S}$ be a σ-algebra of subsets of S. We call $(S, \mathcal{S})$ a *measurable vector space* iff both the following hold:

(a) addition is jointly measurable from $S \times S$ into S,

(b) scalar multiplication is jointly measurable from $\mathbf{R} \times S$ into S, for completed Lebesgue measure λ on $\mathbf{R}$.

Let S be a topological vector space and let $\mathcal{T}$ be the σ-algebra of Borel sets (generated by the open sets). Then if S is metrizable and separable, $(S, \mathcal{T})$ is a measurable vector space, but it need not be so in general.

If $(S, \mathcal{S})$ is a measurable vector space and μ and ν are finite, countably additive measures on $\mathcal{S}$, then we have the convolution $\mu * \nu$ defined as usual by

$$(\mu * \nu)(A) = (\mu \times \nu)\{\langle x, y\rangle : x + y \in A\}.$$

For any (S-valued) random-variable Z, let its probability distribution (law), defined on $\mathcal{S}$, be denoted by $\mathcal{L}(Z)$.

Given any vector space S and $c \in \mathbf{R}$, let $m_c(x) \equiv cx$ for all $x \in S$, and $\theta_s(x) = x + s$ for any $s \in S$.

Definition. Given a measurable vector space $(S, \mathcal{S})$, a probability

Received by the editors October 9, 1973.

AMS (MOS) subject classifications (1970). Primary 60B15; Secondary 60E05, 60F20, 28A40.

Key words and phrases. Stable measure, strictly stable, zero-one law, measurable vector space.

[1] This research was partially supported by National Science Foundation Grant GP-29072.

measure μ on $\mathcal{S}$ will be called *stable* iff for any $A > 0$ and $B > 0$, and independent random variables X and Y with distribution μ, there is a $C > 0$ and an $s \in S$ such that

$$\mathcal{L}(C(AX + BY) + s) = \mu, \quad \text{i.e.} \tag{1}$$
$$[(\mu \circ m_A^{-1}) * (\mu \circ m_B^{-1})] \circ m_C^{-1} = \mu \circ \theta_{-s}^{-1}.$$

Definition. We say μ is *strictly* stable if we can always take $s = 0$ in (1). We call μ *symmetric* iff $\mu(-E) = \mu(E)$ for all $E \in \mathcal{S}$. We say μ is stable of *index* γ if we can always take $C = C_\gamma(A, B) \equiv (A^\gamma + B^\gamma)^{-1/\gamma}$.

A random variable will be called stable iff its distribution is stable, and likewise for symmetry, strict stability, and the index.

The zero-one law (Theorem 6 and corollary) and its proof depend only on the above definitions. The following material, however, will help to clarify the meaning of stability in infinite-dimensional spaces.

If $S = \mathbf{R}$, then, as is well known (cf. Loève [4, pp. 326–328]) every stable μ has some index $\gamma \in (0, 2]$, and the characteristic functions of stable laws of index γ are of the form $e^{\phi(u)}$ where

$$\begin{aligned} \phi(u) &= i\alpha u - b|u|^\gamma\{1 + ic(\operatorname{sgn} u)\tan(\pi\gamma/2)\}, \quad \gamma \neq 1; \\ &= i\alpha u - b|u|\{1 + ic(\operatorname{sgn} u)2\pi^{-1}\log|u|\}, \quad \gamma = 1, \end{aligned} \tag{2}$$

where $\alpha \in \mathbf{R}$, $b > 0$, $|c| \le 1$, $\gamma \in (0, 2]$.

Simple calculations show that a stable μ with characteristic function having logarithm (2) is strictly stable iff $\alpha = 0$ for $\gamma \neq 1$, while for $\gamma = 1$ we have instead $c = 0$. Also, μ is symmetric iff $\alpha = c = 0$. μ has a finite mean iff $\gamma > 1$ (Feller [1, Vol. II., Theorem 1, p. 576]). Then the mean is 0 iff μ is strictly stable. Thus a strictly stable law may be called *centered* stable.

Theorem 1. *Let $(S, \mathcal{S})$ be a measurable vector space and μ a symmetric stable measure on it. Then $\mu \circ \theta_{2s}^{-1} = \mu$ for every s appearing in* (1).

Before proving this we note that in many cases it implies $s = 0$, i.e. μ is strictly stable. For example if S is a complete separable metric linear space and $\mathcal{S}$ is the Borel sets, s must be 0. To prove that every symmetric stable μ is strictly stable it would suffice to show that $\mu \circ \theta_{2s}^{-1} = \mu$ implies $\mu \circ \theta_s^{-1} = \mu$, but we do not know whether this is true in general.

If $\mathcal{S}$ is a "degenerate" σ-algebra, e.g. if it is the smallest σ-algebra for which one linear form ϕ is measurable, and S is more than one-dimensional, there exist $s \neq 0$ and μ such that $\mu \circ \theta_{us}^{-1} = u$ for all $u \in \mathbf{R}$.

Proof. Suppose (1) holds for some s. Then in the notation of the definition of stability, and letting $\mathcal{L}(Z)$ denote the distribution of Z, we have

$$\mathcal{L}(X - s) = \mathcal{L}(C(AX + BY)) = \mathcal{L}(-C(AX + BY)) = \mathcal{L}(-X + s) = \mathcal{L}(X + s),$$

so the conclusion follows.

If μ is a stable law on a measurable vector space $(S, \mathcal{S})$ and ϕ is an $\mathcal{S}$-measurable linear form on S, then $\mu \circ \phi^{-1}$ is clearly stable on **R**. If μ has index γ, so does $\mu \circ \phi^{-1}$. If μ is strictly stable or symmetric, $\mu \circ \phi^{-1}$ has the same property.

It is known that on a complete metric linear space, any Borel measurable or even universally measurable linear form is continuous (theorems of Banach and Douady; cf. L. Schwartz [5, Lemme 2]). There exist such spaces, e.g. $L^p([0, 1], \lambda)$ for $0 \le p < 1$, without any nonzero continuous linear forms, hence without nonzero measurable linear forms. On the other hand if X_n are independent strictly stable real random variables of the same index γ and f_n are functions in L^p with $\int |f_n|^p \, d\lambda \to 0$ fast enough as $n \to \infty$, then $\Sigma X_n f_n$ almost surely converges to an L^p-valued random variable whose distribution is clearly strictly stable of index γ. Thus stable measures on infinite dimensional spaces cannot always be treated in terms of characteristic functions nor measurable linear forms.

2. **Linear forms.** Let S be a real vector space and T a vector space of linear forms on S. Let $\mathcal{S}(T)$ be the smallest σ-algebra for which all members of T are measurable.

Theorem 2. *Let S be a vector space and F a vector space of linear forms on S. Then $(S, \mathcal{S}(F))$ is a measurable vector space.*

Let μ be a probability measure on $\mathcal{S}(F)$. Then μ is strictly stable iff $\mu \circ t^{-1}$ is strictly stable for all $t \in F$.

Proof. To show that $(S, \mathcal{S}(F))$ is a measurable vector space it is enough to show that for each $f \in F$, the maps $\langle s, t\rangle \to f(s + t)$ and $\langle x, s\rangle \to f(xs)$ are jointly measurable, which they clearly are.

As to the stability part, "only if" is clear, as above. To prove "if", suppose each $\mu \circ t^{-1}$ is strictly stable on **R**. Let $\gamma(t)$ denote the index of t, i.e. $\mu \circ t^{-1}$. This is uniquely determined unless $\mu \circ t^{-1}$ is an atom at 0, and then t is stable of every index; we define $\gamma(t) = 2$ in this case.

For any $d \in (0, 2]$ let $F_d = \{t \in F : \gamma(t) \ge d\}$. Clearly $\gamma(t) = \gamma(ut)$ for any real $u \ne 0$. Hence if $t \in F_d$ then $ut \in F_d$ for all real u.

Suppose $t \in F_d$ and $\tau \in F_d$. Let $X_1, X_2, \cdots$ be independent with distribution μ, and $S_n = X_1 + \cdots + X_n$. Then $\mathcal{L}(t(S_n)/n^{1/\gamma(t)}) = \mathcal{L}(t(X_1))$, and likewise for τ.

For any $\delta < d$, we have $(t + \tau)(S_n)/n^{1/\delta} \to 0$ in probability as $n \to \infty$. Thus either $\gamma(t + \tau) \geq d$ or $t + \tau = 0$ a.s. (μ), in which case we have set $\gamma(t + \tau) = 2 \geq d$. Hence F_d is a linear subspace of F.

For any two-dimensional subspace A of F, $\gamma_A \equiv \{\gamma(t): t \neq 0,\ t \in A\}$ contains at most two points, since otherwise for some $t,\ \tau \in A$, $\gamma(t + \tau) < \gamma(t) < \gamma(\tau)$, contradicting the above for $d = \gamma(t)$.

Further, if γ_A contains two points $\delta < \kappa$, we can only have $\gamma = \kappa$ on a 1-dimensional subspace $B = F_\kappa \cap A$ of A. Let $t \in B$, $t \neq 0$, and $t(n) \in A \sim B$, $t(n) \to t$ in the usual topology of a plane. Then for any $u \in \mathbf{R}$,

$$\lim_{n\to\infty} |Ee^{iut(n)}|^2 = |Ee^{iut}|^2.$$

From (2) above we get

$$|Ee^{iut(n)}|^2 = \exp(-c(n)|u|^\delta)$$

for some $c(n) > 0$, so we must have $c(n) \to 0$ and $t = 0$ a.s. (μ).

Since any two points of F belong to some two-dimensional space, it follows that for some $\gamma \in (0, 2]$, all $t \in F$ are strictly stable of index γ; this includes the possibility $t = 0$ a.s.

Now we show μ is strictly stable of index γ.

Let $C = C_\gamma(A, B)$ and $\nu = \mathcal{L}(C(AX + BY))$ in (1). Since every t is strictly stable of index γ, we have $\nu \circ t^{-1} = \mu \circ t^{-1}$, and hence $\int e^{it}\,d\nu = \int e^{it}\,d\mu$.

By the uniqueness theorem for characteristic functions on finite-dimensional vector spaces, the joint distribution of any finite set $(t_1, \cdots, t_n) \subset F$ is the same for ν as for μ. Now since ν and μ are defined on $\mathcal{S}(F)$, they are equal, i.e. (1) holds. Q.E.D.

Theorem 3. *Let S be a vector space and F a vector space of linear forms on S, and μ a probability measure on $\mathcal{S}(F)$. Then μ is symmetric iff $\mu \circ t^{-1}$ is symmetric for all $t \in F$; also μ is symmetric and stable iff $\mu \circ t^{-1}$ is symmetric and stable for all $t \in F$.*

Proof. This is an easy application of the previous theorem and method of proof. □

Definition. Given a real vector space S and a vector space F of linear forms on S, we say (S, F) is a *full* pair iff every real linear form ϕ on F can be written $\phi(f) \equiv f(s)$ for some $s = s_\phi \in S$.

An example of a full pair is $(\mathbf{R}^T, F)$ where $\mathbf{R}^T$ is the set of all real-valued functions on a set T and F is the set of finite linear combinations of coordinate evaluations.

Theorem 4. *Suppose* (S, F) is a *full pair and* μ *is a probability measure on* $(S, \mathfrak{S}(F))$ *such that* $\mu \circ f^{-1}$ *is stable for all* $f \in F$. *Then* μ *is stable of some index* $\gamma \in (0, 2]$, *and if* $\gamma \neq 1$ *there is a unique* $t \in S$ *such that* $\mu \circ \theta_t^{-1}$ *is strictly stable.*

Proof. Let $\mu^-(E) \equiv \mu(-E)$ and let $\nu = \mu * \mu^-$. Then for every $f \in F$, $\nu \circ f^{-1}$ is symmetrically stable. Thus by Theorems 2 and 3, ν is symmetric and strictly stable of some index $\gamma \in (0, 2]$. It follows that for each $f \in F$, $\mu \circ f^{-1}$ is stable of index γ.

For each $f \in F$ there is some real number $k(f)$ such that if X and Y are independent with distribution μ, then for $C = C_\gamma(A, B)$,

$$\mathfrak{L}(f(C[AX + BY]) + k(f)) = \mathfrak{L}(f(X)). \tag{3}$$

Since a Borel probability measure on $\mathbf{R}$ cannot equal a translate of itself, $k(\cdot)$ must be a linear form on F. By fullness there is some $s = s(A, B) \in S$ such that $k(f) = f(s)$ for all $f \in F$. As in the previous proof, it follows using characteristic functions that (1) holds, so μ is stable.

Now suppose $\gamma \neq 1$. For any $f \in F$, there is some unique number α such that $f(X) + \alpha$ is strictly stable, by the characterization of strictly stable laws in (2). If we let $t = 2^{1/\gamma}s(1, 1)/(2 - 2^{1/\gamma})$, then we must have $\alpha = f(t)$ by (1) with $A = B = 1$. Hence $f(X + t)$ is strictly stable. Thus by Theorem 2, $\mu \circ \theta_t^{-1}$ is strictly stable. Q.E.D.

If (S, F) is not a full pair, then at any rate there is a topology $\mathfrak{T}(S)$ on F such that $(F, \mathfrak{T})$ is a topological vector space and a linear form ϕ on F is $\mathfrak{T}$-continuous iff $\phi(f) \equiv f(s)$ for some $s \in S$. The weakest such topology is the weakest topology making each $f \to f(s)$ continuous; there may be others, as with infinite-dimensional Banach spaces and their duals.

Now let f_n be a *sequence* such that $f_n \to 0$ for $\mathfrak{T}$. Then $f_n \to 0$ pointwise on S, so in (3) we must have $k(f_n) \to 0$. Hence k is sequentially $\mathfrak{T}$-continuous.

Definition. We call (S, F) a *semifull pair* iff every sequentially $\mathfrak{T}(S)$-continuous linear form on F is of the form $f \to f(s)$ for some $s \in S$.

For examples of semifull pairs, let $(F, \mathfrak{U})$ be any metrizable linear space and S the dual space of all continuous linear forms on F.

Theorem 5. = *Theorem 4 with "full" replaced by "semifull".*

3. The zero-one law for linear subspaces.

Lemma 1. *Let* μ *be a strictly stable measure on* $(S, \mathfrak{S})$. *Let* E *be a* μ *completion measurable linear subspace of* S. *Let* $E' = \{x | x \in S,\ E - x/r$ *is*

μ completion measurable for all rational numbers $r > 0\}$. Then E' is completion measurable and $\mu(E') = 1$.

Proof. For every rational $r > 0$, there is a positive real number $t(r)$ such that $\mu \circ \mu \circ m_r^{-1} = \mu \circ m_{t(r)}^{-1}$ by (1). Now $E/t(r) = E$ is μ measurable, hence E is $\mu \circ m_{t(r)}^{-1}$ measurable. So for every r, $\exists$ sets F_r, G_r in $\mathcal{S}$ with $F_r \subset E \subset G_r$ and $\mu \circ m_{t(r)}^{-1}(G_r \sim F_r) = 0$. Let $F = \bigcup F_r$, $G = \bigcap G_r$, then we still have $\mu \circ m_{t(r)}^{-1}(G \sim F) = 0$ for all r.

We can write

$$\mu \circ m_{t(r)}^{-1}(G \sim F) = \int_S \mu \circ m_r^{-1}((G - x) \sim (F - x))d\mu(x)$$
$$= \int_S \mu((G/r - x/r) \sim (F/r - x/r))d\mu(x).$$

Let $C_{r,x} = (G/r - x/r) \sim (F/r - x/r)$. It follows that for all r, $\mu\{x| \mu(C_{r,x}) = 0\} = 1$. Define $E'' = \{x|\mu(C_{r,x}) = 0$ for all rational $r\}$ and conclude that E'' is in $\mathcal{S}$ and $\mu(E'') = 1$. However, since $F/r - x/r \subset E/r - x/r \subset G/r - x/r$, we conclude that $E'' \subset E'$ and that E' is μ completion measurable with $\mu(E') = 1$. Q.E.D.

Definition. If μ is strictly stable, we shall say that μ is well behaved if for all $\alpha \in (0, 1)$ there is a $\beta \in (0, 1)$ with

$$(\mu \circ m_\alpha^{-1}) * (\mu \circ m_\beta^{-1}) = \mu. \tag{4}$$

It is easy to show that if μ is strictly stable of index γ then it is well behaved.

Theorem 6. *Let μ be strictly stable and well behaved. Let E be a linear subspace of S, measurable for the completion of μ. Then $\mu(E) = 0$ or 1.*

Proof. Suppose $\mu(E) > 0$. Take α and β so that (4) is satisfied with α rational. Let E' be defined as in the last lemma. Suppose $x \in E' \sim E$ with $\mu(E - x) > 0$. Then

$$\mu(E - \alpha x) = [(\mu \circ m_\alpha^{-1}) * (\mu \circ m_\beta^{-1})](E - \alpha x)$$
$$\geq (\mu \circ m_\alpha^{-1})(E - \alpha x)(\mu \circ m_\beta^{-1})(E) = \mu(E - x)\mu(E) > 0.$$

So the cosets $E - \alpha x$ are all disjoint with measure bounded away from zero, as α ranges over the rationals. This is a contradiction so we conclude that $\mu(E - x) = 0$ for $x \in E' \sim E$.

Now take $A > 0$ and take $C = C(A, A)$ so that (1) holds with $A = B$. Letting $\delta = AC$, we have that (4) holds with $\alpha = \beta = \delta$. We compute

$$\mu(E) = (\mu \circ m_\delta^{-1}) * (\mu \circ m_\delta^{-1})(E).$$

By arguing as in the preceding lemma we can conclude that $E - x$ is $\mu \circ m_\delta^{-1}$ completion measurable for $\mu \circ m_\delta^{-1}$ almost all x and we can write

$$\mu(E) = \int_S \mu((E - \delta x)/\delta)d\mu(x)$$
$$= \int_{E'} \mu(E - x)d\mu(x) = \int_E \mu(E - x)d\mu(x) = \mu(E)^2.$$

Thus, $\mu(E) = 0$ or 1. Q.E.D.

Corollary. *If E is a linear subspace of S with $E \in \mathcal{S}$, then for any stable measure μ we have $\mu(E) = 0$ or 1.*

Proof. Define the measure $\bar{\mu}$ by setting $\bar{\mu}(G) = \mu(-G)$ for $G \in \mathcal{S}$. Define ν, the symmetrization of μ, by $\nu = \mu * \bar{\mu}$. It is clear that ν is strictly stable since for all $A, B > 0$, $\bar{\mu}$ satisfies (1) with $-s$ substituted for s. Now $\nu(E) \geq \mu(E)^2$. If $\mu(E) > 0 < \nu(E + x)$, $x \notin E$, let $T = \{\alpha: \nu(E + \alpha x) \geq \nu(E)\nu(E + x) > 0\}$. Then for any A, B, C in (1), CA and CB are in T as in Theorem 6. Thus the set of ratios of elements of T is infinite and so is T. So we conclude that $\nu(E) = 1$ as in the end of Theorem 6. But

$$\nu(E) = \int \mu(E + x)d\mu(x).$$

It follows that $\mu(E + x) = 1$ for μ almost all x in S. In particular $\mu(E + x) = 1$ for at least one x in E since $\mu(E) > 0$. Now $E = E + x$ for x in E, so $\mu(E) = 1$. Q.E.D.

We can extend Theorem 6 to sets E in $\mathcal{S}$ which are only assumed to be rational linear subspaces of S (i.e., if $x, y \in E$ then $rx + sy \in E$ for any rational r, s).

Theorem 7. *Suppose μ is stable of index γ, where γ is rational. If E is an $\mathcal{S}$ measurable rational linear subspace of S, then $\mu(E) = 0$ or $\mu(E) = 1$.*

Proof. By arguing as in the corollary to Theorem 6, we are reduced to considering the case when μ is symmetric and strictly stable. We assume $\gamma = p/q$ where p and q are integers. By Waring's theorem [3, p. 37] there exists a positive integer k such that any positive integer b can be written in the form $b = a_1^p + \cdots + a_k^p$, where $a_1, \cdots, a_k$ are nonnegative integers. Now consider the equation

$$a^p = (a-1)^p + a_1^p + \cdots + a_k^p. \tag{5}$$

For any positive integer a, (5) has a positive solution in integers a_i. We define $r_0 = ((a-1)/a)^q$, $r_1 = (a_1/a)^q, \cdots, r_k = (a_k/a)^q$ and we let $F(a) = \Sigma_0^k r_i$. We conclude that $F(a) \to 1$ as a goes to infinity and hence the equation

$$1 = r_0^\gamma + r_1^\gamma + \cdots + r_k^\gamma \tag{6}$$

has infinitely many rational solutions and $\Sigma_0^k r_i$ can assume infinitely many values (unless $\gamma = 1$; then proceed as in Theorem 6).

Now $\mu = \mu \circ m_{r_0}^{-1} * \cdots * \mu \circ m_{r_k}^{-1}$ for any solution of (6) and we compute

$$\mu(E + (r_0 + \cdots + r_k)x) \geq (\mu \circ m_{r_0}^{-1}(E + r_0 x)) \cdots (\mu \circ m_{r_k}^{-1}(E + r_k x))$$
$$= (\mu(E + x))^{k+1}.$$

If x is not in E and $\mu(E + x) > 0$, this leads to a contradiction, so we conclude $\mu(E + x) = 0$ for all $x \notin E$. The same reasoning shows that $\mu \circ m_\alpha^{-1}(E + x) = 0$ for all $x \notin E$ and α rational.

Let now $\alpha > 0$, $\beta > 0$, $\theta > 0$ be real with $\alpha^\gamma + \beta^\gamma = \theta^\gamma$ and α rational. We claim that

$$\mu \circ m_\alpha^{-1}(E)\mu \circ m_\beta^{-1}(E) = \mu \circ m_\theta^{-1}(E). \tag{7}$$

This will follow if we show that the $(\mu \circ m_\alpha^{-1}) \times (\mu \circ m_\beta^{-1})$ product measure of the set $\{(x, y) | x \notin E, y \notin E, x + y \in E\}$ is 0. However, this product measure equals $\int_{E^c} \mu \circ m_\alpha^{-1}(E - y)\, d\mu \circ m_\beta^{-1}(y)$, which clearly is zero by the foregoing.

Now let $r_1, \cdots, r_k$ be any rational solution of (6) and define θ_j by $r_1^\gamma + \cdots + r_j^\gamma = \theta_j^\gamma$ for all $1 \leq j \leq k$. We have

$$\mu \circ m_{\theta_2}^{-1}(E) = \mu \circ m_{r_1}^{-1}(E)\, \mu \circ m_{r_2}^{-1}(E).$$

Proceeding inductively by using (7), we conclude $\mu(E) = \Pi_{1 \leq j \leq k}\, \mu \circ m_{r_j}^{-1}(E)$. However, $\mu(E) = \mu \circ m_{r_j}^{-1}(E)$ for all j so we conclude $\mu(E) = 0$ or 1. Q.E.D.

Acknowledgement. In the proof of Theorem 2 we had a useful discussion with D. Cohn and S. Portnoy.

REFERENCES

1. W. Feller, *An introduction to probability theory and its applications.* Vol. II, 2nd ed., Wiley, New York, 1971. MR 42 #5292.

2. G. Kallianpur, *Zero-one laws for Gaussian processes*, Trans. Amer. Math. Soc. 149 (1970), 199–211. MR 42 #1200.

3. A. Hinčin, *Three pearls of number theory*, Graylock Press, New York, 1952. MR 13, 724.

4. M. Loève, *Probability theory.* 3rd ed., Van Nostrand, Princeton, N.J., 1963. MR 34 #3596.

5. L. Schwartz, *Sur le théorème du graphe fermé*, C. R. Acad. Sci. Paris Sér. A–B 263 (1966), A602–A605. MR 34 #6494.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

DEPARTMENT OF MATHEMATICS, SIR GEORGE WILLIAMS UNIVERSITY, MONTREAL, QUEBEC, CANADA