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A CHARACTERIZATION OF THE CONNECTIVITY OF A MANIFOLD IN TERMS OF LARGE OPEN CELLS

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ABSTRACT. If k and n are integers, $0 \leq k \leq n-3$, and M^n is a topological n-manifold without boundary, it is shown that M is k-connected if and only if there is a "tame" $(n-k-1)$-dimensional closed subset X in M such that $M - X$ is homeomorphic to E^n.

1. **Introduction.** In [1] Morton Brown proved that any compact connected topological n-manifold M^n without boundary is the continuous image of an n-cell such that the boundary of the cell is taken onto a subset of M having dimension $\leq n-1$ and the interior of the cell is taken homeomorphically onto the rest of M. Thus any compact n-manifold can be obtained from euclidean n-space E^n by simply (and carefully) pasting on a space X of dimension $\leq n-1$. We examine here the question of just how small the dimension of X can be made and show that this question is directly related to the connectivity of M.

By a (*topological*) n-*manifold* M^n, we mean a separable metric space, each point of which has an open neighborhood homeomorphic to E^n (we shall assume throughout that all manifolds are *without* boundary). A *euclidean chart* for M is a pair (h, W) where W is an open subset of M and $h: E^n \to W$ is a homeomorphism. For any chart (h, W) of M and any real number $t > 0$, let $W_t = h(C_t)$ where C_t is the (closed) n-cell in E^n with center 0 and radius t.

A closed subset X of a topological space Y is a Z_1-*set* if for every nonempty 1-connected open subset U of Y, $U - X$ is nonempty and 1-connected. If Y is a metric space, X is a subset of Y, and $\epsilon > 0$, let $N(X, \epsilon)$ denote the set of points in Y whose distance from X is less than ϵ. An ϵ-*push* of the pair (Y, X) is a homeomorphism h of Y for which an ϵ-isotopy H of Y exists satisfying: $H_0 = 1$, $H_1 = h$, and $H_t | Y - N(X, \epsilon) = 1$ for each

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t in $[0, 1]$. A closed subset X of E^n is said to be a *strong* Z_m*-set* in E^n (m an integer, $-1 \leq m < n$) if for each compact subpolyhedron Q of E^n having dimension $\leq m + 1$, and each $\epsilon > 0$, there exists an ϵ-push h of $(E^n, X \cap Q)$ such that $h(X) \cap Q = \emptyset$. Heuristically, one should think of a strong Z_m-set in E^n as a "tame" subset of E^n having dimension $\leq n - m - 2$: the strong Z_m-sets in E^n, $m \geq 1$, are precisely those which are Z_1-sets and have dimension $\leq n - m - 2$ (see 2.1 below). A closed subset X of an n-manifold M is a *strong* Z_m*-set* in M if for each point x in X there is a euclidean chart (h, W) for M such that $x \in W$ and $h^{-1}(X)$ is a strong Z_m-set in E^n.

The main result of this paper is

Theorem 1.1. *Let k and n be integers, $0 \leq k \leq n - 3$, and let M^n be an n-manifold. Then M is k-connected if and only if there is a strong Z_{k-1}-set X in M such that $M - X$ is homeomorphic to E^n.*

Our proof in the "if" direction is basically a general position argument: A singular k-sphere in M is pushed off X, and hence into $M - X$, where it bounds. The proof in the "only if" direction is similar to Morton Brown's proof in [1]. We start with an open cell in M and engulf a dense k-dimensional subset of M whose complement is $(n - k - 1)$-dimensional. The complement of the open cell (after the engulfing) is a strong Z_{k-1}-set in M. A more useful form of 1.1 is

Theorem 1.2. *Let M^n be an n-manifold.*

(1) *M is connected if and only if there is a closed subset X of M such that* dim $X \leq n - 1$ *and $M - X$ is homeomorphic to E^n.*

(2) *If $n \geq 4$, then M is simply connected if and only if there is a closed subset X of M such that* dim $X \leq n - 2$ *and $M - X$ is homeomorphic to E^n.*

(3) *If $2 \leq k \leq n - 3$, then M is k-connected if and only if there is a Z_1-set X in M such that* dim $X \leq n - k - 1$ *and $M - X$ is homeomorphic to E^n.*

Throughout the remainder of this paper let M^n be a fixed n-manifold with metric d.

In §2 we shall discuss strong Z_m-sets in M and in §3 we present the proofs of 1.1 and 1.2.

2. Tame sets in topological manifolds.

First we give a characterization of strong Z_m-sets in terms of dimension and local homotopy properties.

Lemma 2.1. *Let m be an integer, $-1 \leq m < n$, and let X be a closed subset of M.*

(1) *If X is a strong Z_m-set in M, then* dim $X \leq n - m - 2$.

(2) *If* $\dim X \le n-1$, *then* X *is a strong* Z_{-1}*-set in* M.

(3) *If* $\dim X \le n-2$ *and* $n \ne 3$, *then* X *is a strong* Z_0*-set in* M.

(4) *If* $\dim X = n-m-2 \le n-3$, $n \ne 4$, *and* X *is a* Z_1*-set in* M, *then* X *is a strong* Z_m*-set in* M.

Proof. For the case $M = E^n$, this lemma is a precise restatement of 3.1 of [5]. To prove the lemma for an arbitrary manifold, one need only look at charts and apply the case $M = E^n$.

Lemma 2.2. *Let* X *be a strong* Z_m*-set in* M *and let* Q *be a compact* $(m+1)$*-dimensional polyhedron. Then any map* $f: Q \to M$ *is homotopic to a map* $g: Q \to M - X$.

Proof. There exist finitely many euclidean charts $(h_1, W_1), \cdots, (h_r, W_r)$ such that $f(Q) \subset \bigcup_{i=1}^r W_i$ and $h_i^{-1}(X)$ is a (possibly empty) strong Z_m-set in E^n. Furthermore, there exists a real number t such that $f(Q) \subset W_{1t} \cup \cdots \cup W_{rt}$ where $W_{it} = h_i(C_t)$ for each $i \le r$. For each $i \le r$, let Q_i be a compact subpolyhedron of Q such that $f^{-1}(W_{it}) \subset Q_i \subset f^{-1}(W_i)$. We shall construct, by induction, a sequence of maps $f_1, \cdots, f_r$ from Q into M such that

(1) f is homotopic to f_1, and f_i is homotopic to f_{i+1} for each $i \le r$,

(2) $f_i(Q_j) \subset W_j$ for each $i, j \le r$, and

(3) $f_i(Q_1 \cup \cdots \cup Q_i) \cap X = \emptyset$ for each $i \le r$.

Then $g = f_r$ is a map homotopic to f and $g(Q) \cap X = \emptyset$. To start the induction consider the map $f|Q_1: Q_1 \to W_1$. By the simplicial approximation theorem there is a map $f_1': Q \to M$ such that f_1' is homotopic to f, $f_1'(Q_j) \subset W_j$ for each $j \le r$, and $f_1'\,|Q_1: Q_1 \to W_1$ is PL where W_1 has the PL structure induced by h_1. Since $h_1^{-1}(X)$ is a strong Z_m-set in E^n, there is a homeomorphism $g_1: M \to M$, isotopic to the identity, such that $g_1 f_1'(Q_j) \subset W_j$ for each $j \le r$ and $g_1 f_1'(Q_1) \cap X = \emptyset$. Then $f_1 = g_1 f_1'$ is homotopic to f, $f_1(Q_j) \subset W_j$ for each $j \le r$, and $f_1(Q_1) \cap X = \emptyset$. Now suppose that f_{i-1} has been chosen, $i \le r$, and consider the map $f_{i-1}|Q_i: Q_i \to W_i$. By the simplicial approximation theorem there is a map $f_i': Q \to M$ such that f_i' is homotopic to f_{i-1}, $f_i'(Q_j) \subset W_j$ for each $j \le r$, $f_i'(Q_1 \cup \cdots \cup Q_{i-1}) \cap X = \emptyset$, and $f_i'\,|Q_i: Q_i \to W_i$ is PL where W_i has the structure induced by h_i. Since $h_i^{-1}(X)$ is a strong Z_m-set in E^n, there is a homeomorphism $g_i: M \to M$, isotopic to the identity, such that $g_i f_i'(Q_j) \subset W_j$ for each $j \le r$, g_i is fixed on $f_i'(Q_1 \cup \cdots \cup Q_{i-1})$, and $g_i f_i'(Q_i) \cap X = \emptyset$. Then $f_i = g_i f_i'$ is homotopic to f, $f_i(Q_j) \subset W_j$ for each $j \le r$, and $f_i(Q_1 \cup \cdots \cup Q_i) \cap X = \emptyset$.

We now construct a k-dimensional dense subset of E^n. Let k be an integer, $0 \le k < n$, and let J_0 be a rectilinear PL triangulation of E^n such that all the n-simplexes of J_0 have the same diameter. For each integer

$i \geq 1$, let J_i be the ith barycentric subdivision of J_0 and let J_i^k be the k-skeleton of J_i. Finally, set $\tilde{B}_n^k = \bigcup_{i=1}^{\infty} |J_i^k|$ and $\tilde{P}_n^{n-k-1} = E^n - \tilde{B}_n^k$. Clearly $\tilde{B}_n^k$ is k-dimensional and $\tilde{P}_n^{n-k-1}$ is $(n-k-1)$-dimensional. Moreover, $\tilde{B}_n^k$ satisfies a very nice "absorption" property:

Lemma 2.3. *Let Q be a compact k-dimensional subpolyhedron of E^n, let U be an open subset of E^n, and let $\epsilon > 0$. Then there is a homeomorphism $h: E^n \to E^n$, fixed outside U and moving points a distance less than ϵ, such that $h(Q \cap U) \subset \tilde{B}_n^k$.*

Proof. This follows directly from Lemma 4.5 of [5].

Lemma 2.4. *A closed subset X of E^n which is contained in $\tilde{P}_n^{n-k-1}$ is a strong Z_{k-1}-set in E^n.*

Proof. Let Q^k be a compact k-dimensional subpolyhedron of E^n, let U be an open set in E^n containing $X \cap Q$, and let $\epsilon > 0$. By 2.3, there is a homeomorphism h, fixed outside U and moving points a distance less than ϵ, such that $h(Q \cap U) \subset \tilde{B}_n^k$. In particular, $h^{-1}(X) \cap Q = \emptyset$. Hence if h^{-1} were ϵ-isotopic to the identity by an isotopy fixing $E^n - U$, then X would be a strong Z_{k-1}-set. But the existence of such an isotopy follows easily from the results in [3].

We now construct a dense k-dimensional subset of M. Let $\{(h_i, W_i)\}$ be a countable collection of euclidean charts such that $M = \bigcup_{i=1}^{\infty} W_i$ and let $\tilde{B}_M^k = \bigcup_{i=1}^{\infty} h_i(\tilde{B}_n^k)$ and $\tilde{P}_M^{n-k-1} = M - \tilde{B}_M^k$. Clearly $\tilde{B}_M^k$ is k-dimensional and $\tilde{P}_M^k$ is $(n-k-1)$-dimensional. Moreover, $\tilde{B}_M^k$ can be written as the countable union of compact subsets $\{\tilde{B}_{(i)}^k\}_{i=1}^{\infty}$ of M having the following property: if $i \geq 1$, then there exists $j \geq 1$ such that $\tilde{B}_{(i)}^k \subset W$, and $h_j^{-1}(\tilde{B}_{(i)}^k)$ is a compact subpolyhedron of E^n having dimension $\leq k$. *For the remainder of this paper we fix $\{(h_i, W_i)\}$, $\tilde{B}_M^k$, $\tilde{P}_M^{n-k-1}$, and $\{\tilde{B}_{(i)}^k\}$ as above.*

The final lemma of this section follows directly from 2.4.

Lemma 2.5. *A closed subset of M which is contained in $\tilde{P}_M^{n-k-1}$ is a strong Z_{k-1}-set in M.*

3. Connectivity in topological manifolds. Let X be a Z_{k-1}-set in M where $0 \leq k < n$ and let $f: S^k \to M$ be a map of the k-sphere into M. By Lemma 2.2, f is homotopic to a map $g: S^k \to M - X$. If $M - X$ is homeomorphic to E^n, then g extends to a map of the $(k+1)$-ball into $M - X$ and hence f is null-homotopic in M. This proves

Theorem 3.1. *If there is a strong Z_{k-1}-set X in M^n, $0 \leq k < n$, such that $M - X$ is homeomorphic to E^n, then M is k-connected.*

To prove the converse of 3.1 (for codimension ≥ 3), we require the following engulfing lemma; its proof is almost precisely the same as the proof of Lemma 1 of [2] and therefore we leave the details to the reader.

Lemma 3.2. *Let k be an integer, $0 \leq k \leq n-3$, and let M^n be k-connected. Let Q be a compact subset of M such that for some chart (g, U) of M, $Q \subset U$ and $g^{-1}(Q)$ is a k-dimensional subpolyhedron of E^n. If (h, W) is a euclidean chart for M and t is a positive real number, then there is a homeomorphism f of M such that $f|W_t = 1$ and $f(W) \supset Q$.*

Lemma 3.3. *If k is an integer, $0 \leq k \leq n-3$, and M^n is k-connected, then there is a euclidean chart (g, U) of M such that U contains $\tilde{B}^k_M$.*

Proof. Consider the compact subsets $\tilde{B}^k_{(i)}$ of M as defined in the previous section and let (h, W) be any euclidean chart of M. By Lemma 3.2, there is a homeomorphism f_1 of M such that $f_1(W) \supset \tilde{B}^k_{(1)}$. Since $\tilde{B}^k_1$ is compact, there is a real number $t_1 \geq 1$ such that $f_1(W_{t_1}) \supset \tilde{B}^k_{(1)}$. Applying the lemma again, there is a homeomorphism f_2 of M such that $f_2|f_1(W_{t_1}) = 1$ and $f_2 f_1(W) \supset \tilde{B}^k_{(2)}$. Let $t_2 \geq \max\{t_1, 2\}$ and such that $f_2 f_1(W_{t_2}) \supset \tilde{B}^k_{(2)}$. Continuing inductively, there is a sequence $\{f_i\}$ of homeomorphisms of M and a sequence $t_1 \leq t_2 \leq \cdots$ of real numbers such that

(1) $f_i | f_{i-1} \cdots f_1(W_{t_{i-1}}) = 1$,

(2) $f_i \cdots f_1(W_{t_i}) \supset \tilde{B}^k_{(i)}$, and

(3) $t_i \geq i$

for each $i \geq 1$. Then $f = \lim_{i \to \infty} f_i \cdots f_1 | W$ is an embedding of W into M such that $f(W) \supset \tilde{B}^k_M$ and hence $(g, U) = (fh, f(W))$ is a euclidean chart satisfying the desired condition.

Theorem 3.4. *If k is an integer, $0 \leq k \leq n-3$, and M^n is k-connected, then there is a strong Z_{k-1}-set X in M such that $M - X$ is homeomorphic to E^n.*

Proof. Let (g, U) be a chart in M with $\tilde{B}^k_M \subset U$. Then $X = M - U$ is closed in M and is contained in $\tilde{P}^{n-k-1}_M$. By 2.5, X is a strong Z_{k-1}-set in M.

Proof of Theorem 1.1. Apply 3.1 and 3.4.

Proof of Theorem 1.2. (1) If M is connected, the result follows from the proof of the Theorem in [1]. The proof given there is an elementary form of engulfing and does not require codimension three. The converse is trivial since 0-spheres can easily be pushed off codimension 1 subsets. The proofs of (2) and (3) follow easily from 3.4 and 2.1.

We end the paper with a rather strange corollary to 3.3. While not directly related to the above results, it does show that the subset $\tilde{B}^k_M$ of M is independent of the charts $\{(h_i, W_i)\}$ and, in fact, depends only on the connectivity of M.

Corollary 3.5. *Let k be an integer, $0 \leq k \leq n - 3$, $n \neq 4$, and let M be k-connected. Then $\tilde{B}^k_M$ and $\tilde{B}^k_n$ are homeomorphic.*

Proof. Let (g, U) be a chart in M such that $\tilde{B}^k_M \subset U$. By 5.4 and 2.5 of [5], there is a homeomorphism of E^n which takes $g^{-1}(\tilde{B}^k_M)$ onto $\tilde{B}^k_n$.

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