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# SEPARABLE MENGER-REGULAR HAUSDORFF CURVES ARE METRIZABLE

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ABSTRACT. It is shown that every connected, separable, locally compact and locally peripherally finite Hausdorff space is metrizable.

1. **Introduction.** A Menger-regular Hausdorff space is a connected Hausdorff topological space with a base for the topology consisting of neighborhoods with finite boundaries. We prove the following:

**Theorem 1.** *If $M$ is a Menger-regular Hausdorff continuum and $A$ is a closed subset of $M$, and $S$ is a dense subset of $M$, then there is a subset $T$ of $A$ which is dense in $A$ such that the cardinality of $T$ is less than or equal to the cardinality of $S$.*

**Corollary 1.1.** *If $M$ is a separable Menger-regular Hausdorff continuum, every arc contained in $M$ is metrizable.*

**Theorem 2.** *Every separable Menger-regular Hausdorff continuum is metrizable.*

**Corollary 2.1.** *Every separable locally compact Menger-regular Hausdorff space is metrizable.*

Suppose $M$ is a Menger-regular Hausdorff continuum. Using techniques directly analogous to those for metric continua, it can be shown that each two closed disjoint subsets of $M$ are separated in $M$ by a finite subset of $M$ and that $M$ is locally connected [1, p. 96], [2, p. 107]. Obviously each subcontinuum of $M$ is a Menger-regular Hausdorff continuum and therefore is locally connected. If $O$ is a connected open subset of $M$, and $P$ and $Q$ belong to $O$, there is in $O$ an irreducible continuum $K$ from $P$ to $Q$. Since $K$ is locally connected, each point of $K$ distinct from $P$ and $Q$ is a cutpoint

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of $K$ and $K$ is therefore a Hausdorff arc. Consequently $M$ is Hausdorff arcwise connected and locally Hausdorff arcwise connected. If $P$ is a point of $M$ and $C$ is a closed subset of $M$, an *arc from* $P$ *to* $C$ *in* $M$ is $\{P\}$ if $P$ belongs to $C$ and if $P$ does not belong to $C$, it is a Hausdorff arc with $P$ as one endpoint, its other endpoint in $C$ and which contains no other point of $C$. We let $N$ denote the positive integers.

2. **Proof of Theorem 1.** Suppose the hypothesis. Let $\omega$ denote the family of all $W$ such that $W$ is a disjoint collection each member of which is an arc from a point of $S$ to $A$. Then $\omega$ is partially ordered by containment, the union of any chain of elements of $\omega$ is a member of $\omega$ which is an upper bound of that chain, and therefore there is a maximal element $W_{\max}$ of $\omega$.

Let $T = A \cap [\bigcup W_{\max}]$. The cardinality of $T$ is obviously less than or equal to the cardinality of $S$. We will show that $T$ is dense in $A$.

Suppose $T$ is not dense in $A$ and that $P$ is a point of $A - \mathrm{Cl}(T)$. There is a finite set $F$ separating $P$ from $\mathrm{Cl}(T)$ in $M$, and because the members of $W_{\max}$ are disjoint, $F$ intersects at most a finite number of them. Consequently, $P$ is not a limit point of $\bigcup W_{\max}$ and there exists a connected open set $O$ that contains $P$ and misses $\bigcup W_{\max}$. Then $O$ contains a point $Q$ of $S$ and also contains an arc $\alpha$ from $Q$ to $A$. But $W_{\max} \cup \{\alpha\}$ is a member of $\omega$ which properly contains $W_{\max}$ and this is a contradiction.

**Proof of Corollary 1.1.** If $\alpha$ is an arc in $M$ and $M$ is separable, then, from Theorem 1, $\alpha$ is separable, and it is known that every separable Hausdorff arc is metrizable [2, p. 31].

3. **Proof of Theorem 2.** Suppose $M$ is a separable Menger-regular Hausdorff continuum and $\{P_0, P_1, P_2, \cdots\}$ is a countable sequence of distinct points which is dense in $M$. We define inductively the sequence $\{A_0, A_1, A_2, \cdots\}$.

(i) Let $A_0$ denote $\{P_0\}$.

(ii) Suppose $n$ is a nonnegative integer and $A_i$ has been defined for $0 \leq i \leq n$. Let $A_{n+1}$ denote an arc from $P_{n+1}$ to $\bigcup \{A_i | i = 0, 1, \cdots, n\}$.

For each $n \geq 0$, let $D_n$ denote $\bigcup \{A_i | i = 0, 1, \cdots, n\}$, and let $Q_{n+1}$ denote the point of $A_{n+1}$ that belongs to $D_n$. Then for $n \in N$, $D_n$ is the union of a finite number of metric arcs and is itself a metric space, in fact, a metric dendron (locally connected metric continuum which contains no simple closed curve). For $n \geq 0$, define $\rho_{n+1}: D_{n+1} \to D_n$ by

$$\begin{aligned} \rho_{n+1}(x) &= x && \text{if } x \in D_n, \\ &= Q_{n+1} && \text{if } x \in A_{n+1}. \end{aligned}$$

It is apparent that, for $n \in N$, $\rho_n$ is continuous, and we let $D_\infty$ denote the inverse limit space of the inverse limit system $\{D_n, \rho_n, n \in N\}$. Since, for $n \in N$, $D_n$ is a compact metric continuum, $D_\infty$ is a compact metric continuum. It can be shown that $D_\infty$ is a dendron, but we do not require this.

Suppose $(x_1, x_2, \cdots)$ belongs to $D_\infty$. For $n \in N$, either $x_{n+1}$ belongs to $D_n$, in which case $x_{n+1} = x_n$, or $x_{n+1}$ does not belong to $D_n$, in which case $x_{n+1}$ belongs to $A_{n+1}$ and $x_n$ is $Q_{n+1}$. In the first case we let $[x_n, x_{n+1}]$ denote $\{x_n\}$, and in the second case we let $[x_n, x_{n+1}]$ denote the subarc of $A_{n+1}$ with endpoints $x_n$ and $x_{n+1}$. For $m, n \in N$, $m < n$, we let $[x_m, x_n]$ denote $\bigcup \{[x_k, x_{k+1}] \mid k = m, n-1\}$. By induction arguments which we omit, it can be shown that if $i, j, k \in N$, $i < j < k$, then

(a) $[x_i, x_j]$ is either $\{x_i\}$ or an arc with endpoints $x_i$ and $x_j$.

(b) $x_i$ is the only point of $[x_i, x_j]$ which belongs to $D_i$.

(c) $x_j$ is the only point common to $[x_i, x_j]$ and $[x_j, x_k]$.

**Lemma 1.** *If $(x_1, x_2, \cdots)$ is a point of $D_\infty$, there is a point $P$ of $M$ such that* $\lim [x_n, x_{n+1}] = \{P\}$.

**Proof.** The sequence $x_1, x_2, \cdots$ has at least one cluster point, $P$. Suppose $q_1, q_2, \cdots$ is a sequence such that, for $n \in N$, $q_n \in [x_n, x_{n+1}]$, and $q_1, q_2, \cdots$ has a cluster point $Q$ distinct from $P$. There is a finite subset $F$ of $M$ such that $M - F = U \cup V$, $U$ and $V$ disjoint open subsets of $M$ containing $P$ and $Q$ respectively. There is an infinite increasing sequence $n(1)$, $n(2), \cdots$ such that for $k \in N$, $x_{n(2k-1)}$ belongs to $U$ and $q_{n(2k)}$ belongs to $V$. Then for $k \in N$, $F$ contains a cutpoint of $[x_{n(2k-1)}, x_{n(2k+1)}]$, and because $F$ is finite, there is a point $f$ of $F$ and there are two integers $i$ and $j$ such that $f$ is a cutpoint of both $[x_{n(2i-1)}, x_{n(2i+1)}]$ and $[x_{n(2j-1)}, x_{n(2j+1)}]$. This and statement (c) above lead to a contradiction. It follows that $\{P\}$ is $\lim [x_n, x_{n+1}]$.

We define $\phi: D_\infty \to M$ by: If $x = (x_1, x_2, \cdots)$ belongs to $D_\infty$, $\phi(x) = \lim x_n$.

It follows from Lemma 1 that $\phi$ is a function. We show that

**Lemma 2.** *$\phi$ is continuous.*

**Proof.** Suppose $x \in D_\infty$ and $O$ is an open set in $M$ containing $\phi(x)$. There is an open set $R$ in $M$ which contains $\phi(x)$ such that $\mathrm{Cl}(R)$ is a subset of $O$ and $R$ has a finite boundary $F$. Since $F$ is finite, there is $i \in N$ such that $D_i$ contains $F \cap [\bigcup \{D_n \mid n \in N\}]$. There is an integer $j$ greater than $i$ such that $x_j$ belongs to $R$ and there is a connected open subset $U_j$ of $D_j$ that contains $x_j$ and is a subset of $R$. Let $U$ denote the points $y$ of $D_\infty$ such that the $j$th coordinate of $y$ belongs to $U_j$. Then $U$ is an open subset of $D_\infty$ which contains $y$ and we next show that $\phi(U) \subset \mathrm{Cl}(R) \subset O$.

Suppose $y = (y_1, y_2, \cdots)$ is a point of $U$ and $\phi(y)$ does not belong to $\mathrm{Cl}(R)$. Then $y_j$ belongs to $U_j$ and therefore belongs to both $D_j$ and $R$. But since $\phi(y) = \lim y_n$ does not belong to $\mathrm{Cl}(R)$, there is an integer $k > j$ such that $y_k$ does not belong to $\mathrm{Cl}(R)$. But then $F$ must contain a point $y'$ of $[y_j, y_k] \subset D_k$, and since $F \cap D_k \subset D_i \subset D_j$, $y'$ belongs to $D_j$. But $y_j$ and $y'$ are two points of $[y_j, y_k]$ that belong to $D_j$ and this is a contradiction to (b) above.

**Lemma 3.** *$\phi$ maps $D_\infty$ onto $M$.*

For $n \in N$, $\phi$ maps the point $(\rho_1 \cdots \rho_{n-1}(P_n), \rho_2 \cdots \rho_{n-1}(P_n), \cdots, \rho_{n-1}(P_n), P_n, P_n, \cdots)$ onto $P_n$. Therefore $\phi(D_\infty)$ is dense in $M$ and since $D_\infty$ is compact, $\phi(D_\infty)$ is compact and is therefore $M$.

We have shown that $M$ is the continuous image of a compact metric space, and since $M$ is Hausdorff, it follows that $M$ is metrizable.

**Proof of Corollary 2.1.** Suppose $M$ is a separable locally compact Menger-regular Hausdorff space. The one point compactification of $M$ is a separable Menger-regular Hausdorff continuum, which by Theorem 2 is metrizable, and it follows that $M$ is metrizable.

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