PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

MULTIPLIERS VANISHING AT INFINITY FOR CERTAIN COMPACT GROUPS[1]

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ABSTRACT. We prove for certain compact groups G and $1 < p < \infty$, $p \neq 2$, that there exist operators commuting with left translations on $L^p(G)$ which are compact as operators on $L^2(G)$ but not as operators on $L^p(G)$.

Let G be a compact group and let Γ be the dual object of G, that is, the set of equivalence classes of irreducible unitary representations of G. For each $\gamma \in \Gamma$ we fix a representative D_γ of γ and a Hilbert space $\mathfrak{H}_\gamma$ of dimension d_γ on which $D_\gamma(x)$ acts. With this notation, if $f \in L^1(G)$ we can write the Fourier series of f as

$$f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma \operatorname{tr}(\hat{f}(\gamma) D_\gamma(x))$$

where tr is the ordinary trace and

$$\hat{f}(\gamma) = \int_G f(x) D_\gamma(x^{-1})\, dx$$

is a linear transformation acting on $\mathfrak{H}_\gamma$. (Warning: the notation here is not the same as in [7]: $\hat{f}(\gamma)$ denotes in this paper what Hewitt and Ross call the coefficient operator, cf. [7, (34.3)(a)].)

Following [7] we denote by $\mathfrak{E} = \mathfrak{E}(\Gamma)$ the space consisting those of functions W on Γ such that $W(\gamma)$ is a linear transformation on $\mathfrak{H}_\gamma$ for each $\gamma \in \Gamma$.

Definition 1. *An element $w \in \mathfrak{E}$ is called a multiplier of $L^p(G)$ $(1 \leq p \leq \infty)$ if for every $f \in L^p(G)$ the series $\sum_{\gamma \in \Gamma} d_\gamma \operatorname{tr}(W(\gamma)\hat{f}(\gamma)D_\gamma(x))$, is the Fourier series of an element $T_W f$ of $L^p(G)$. The space of multipliers is denoted by $M_p(G)$.*

We notice that the operators T_W of L^p into L^p are linear and (by the closed graph theorem) continuous. *We can endow therefore $M_p(G)$ with a Banach space norm: the norm of $W \in M_p(G)$ is defined to be the norm of the*

Received by the editors January 30, 1973.

AMS (MOS) subject classifications (1970). Primary 43A15, 43A75, 22C05.

[1] Work supported by the C.N.R.

corresponding operator T_W *on* $L^p(G)$. It is easy to verify, for $p < \infty$, that the operators T_W, with $W \in M_p$ are exactly the bounded linear operators on $L^p(G)$ which commute by left translations.

Definition 2. *A multiplier* $W \in M_p(G)$ *is said to vanish at infinity if* $\lim_{\gamma\to\infty} \|W(\gamma)\| = 0$, *where the norm is that of* $W(\gamma)$ *as an operator on* $\mathfrak{H}_\gamma$.

It is known that if G is an Abelian group and $p \neq 2$, there exist multipliers which vanish at infinity and are not the limit, in the norm of M_p, of multipliers with finite support. For $p = 1$ or $p = \infty$ this is equivalent to the classical result which asserts the existence of singular measures with Fourier-Stieltjes transform vanishing at infinity. For $1 < p < \infty$, $p \neq 2$, this was proved in [4].

We remark that multipliers which are the limit in $M_p(G)$ of finitely supported ones are precisely those for which the corresponding operator on $L^p(G)$ is compact and that the elements of $\mathfrak{E}(\Gamma)$ which vanish at infinity correspond to compact operators on $L^2(G)$.

The purpose of the present paper is to extend the results described above, when $1 < p < \infty$ and $p \neq 2$, to a class of noncommutative compact groups. We shall prove in fact the following:

Theorem A. *Let* J *be an infinite index set and let* $G = \Pi_{i \in J} G_i$, *where for each* i, G_i *is a nontrivial compact group. Let* $1 < p < \infty$ *and* $p \neq 2$. *Then there exists a multiplier* $W \in M_p(G)$ *which vanishes at infinity and is not the limit, in the norm of* M_p, *of finitely supported elements of* $\mathfrak{E}$.

The proof of this theorem is based on two lemmas. The first lemma is due to C. Fefferman and H. S. Shapiro [2, Theorem 1] and the second to A. Bonami [1, pp. 374–375]. Both lemmas were stated and proved only for commutative compact groups, but proofs can be easily translated into the language of noncommutative groups, as will be indicated below.

Lemma 1 (Fefferman and Shapiro). *Let* $1 < p < \infty$, *then there exists a constant* $\alpha = \alpha(p) > 0$ *such that if* $W \in M_p(G)$, *and* W *satisfies the conditions*: (i) $W(\gamma_0) = 0$, *where* γ_0 *is the equivalence class of the trivial representation*, (ii) $\|W\|_{M_p} \leq \alpha(p)$; *then the multiplier defined by* $W'(\gamma_0) = I$ (*the identity operator*), *and* $W'(\gamma) = W(\gamma)$ *for* $\gamma \neq \gamma_0$, *has norm one.*

The proof of this lemma is almost exactly the same as that which is given in [2] for the corresponding result for commutative groups. Only two remarks are needed. First of all the *norm-decreasing* inclusion $M_p(G) \subseteq M_2(G)$, a well-known fact for G commutative, is a consequence, for noncommutative

G, of recent results of C. Herz [6, Theorem C]. Second, the proof of Theorem 1 in [2] makes use of the fact that $M_p = M_q$, for commutative G, when $1/p + 1/q = 1$. This equality is not known to be true for noncommutative G, but we can use the known fact that $M_p = M_q'$, where M_q' is the space of "right" multipliers of L^q, that is the space of $W \in \mathfrak{S}$ such that for $f \in L^q$, the Four-series $\Sigma_{\gamma \in \Gamma} d_\gamma \operatorname{tr}(\hat{f}(\gamma)W(\gamma)D_\gamma(x))$, represents a function in L^q [3]. With these two remarks in mind the proof of Theorem 1 of [2] is easily reinterpreted to yield Lemma 1.

Before stating the second lemma we remark that if $G = G_1 \times G_2$ where G_1 and G_2 are compact groups with dual objects Γ_1 and Γ_2, respectively, then the dual object Γ of G can be written as $\Gamma = \Gamma_1 \times \Gamma_2$, in the sense that if $\gamma \in \Gamma$, there exists a unique pair $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, such that any representative D_γ of γ is unitarily equivalent to the tensor product $D\gamma_1 \otimes D\gamma_2$ of a representative $D\gamma_1$ of γ_1 and a representative $D\gamma_2$ of γ_2. We shall then write $\gamma = \gamma_1 \times \gamma_2$ [7, Theorem 27.4.3].

Lemma 2 (A. Bonami). *Let G_1 and G_2 be compact groups, with dual objects Γ_1 and Γ_2 and let $G = G_1 \times G_2$. Let W_1 and W_2 be elements of $M_p(G_1)$ and $M_p(G_2)$, respectively, and suppose that $\|W_1\|_{M_p} = \|W_2\|_{M_p} = 1$. (As before γ_0 denotes the class containing the trivial representation.) Then the element $W \in \mathfrak{S}(\Gamma_1 \times \Gamma_2)$ defined by $W(\gamma) = W_1(\gamma_1) \otimes W_2(\gamma_2)$, if $\gamma = \gamma_1 \times \gamma_2$, is an element of $M_p(G)$ and $\|W\|_{M_p} \le 1$.*

Again, the proof of this lemma is exactly as in the commutative case [1, Lemma 1, p. 375].

Proof of Theorem A. If $G = \Pi_{i \in J} G_i$ where G_i are nontrivial groups and J is an infinite set, we may assume, without loss of generality, that each G_i is infinite (if not divide J into infinitely many infinite subsets and group together the factors). We may also assume for simplicity that J is countable and in fact that $J = \{1, 2, \cdots\}$. For each i we know that since $p \neq 2$ the norm-decreasing inclusion $M_p(G_i) \subsetneq M_2(G_i)$ is strict [5, Theorem 6]. This implies that the norm of $M_p(G_i)$ is not equivalent to that of $M_2(G_i)$ because M_p is not closed in M_2. Therefore we can find a finitely supported $W_i \in M_p(G_i)$ such that $W_i(\gamma_0) = 0$, $\|W_i\|_{M_2} \le 1/i$, $\|W_i\|_{M_p} = \alpha(p)$, where $\alpha(p)$ is the constant appearing in the statement of Lemma 1. Let W_i' be the multiplier satisfying $W_i'(\gamma_0) = I$, $W_i'(\gamma) = W_i(\gamma)$ for $\gamma \neq \gamma_0$, whose norm is one by Lemma 1.

Applying inductively Lemma 2 we can construct elements $W^{(n)} \in M_p(G^{(n)})$ where $G^{(n)} = \Pi_{i=1}^n G_i$, such that if $\Gamma^{(n)}$ is the dual object of $G^{(n)}$, and $\gamma \in \Gamma^{(n)}$, $\gamma = \gamma_1 \times \cdots \times \gamma_n$, then $W^{(n)}(\gamma) = W_1'(\gamma_1) \otimes \cdots \otimes W_n'(\gamma_n)$ and $\|W^{(n)}\|_{M_p}$

≤ 1. Obviously we may consider $W^{(n)}$ as an element of $M_p(G)$ with the same norm, by defining $W^{(n)}(\gamma) = 0$, if $\gamma \notin \Gamma_1 \times \cdots \times \Gamma_n$, where Γ_i is the dual object of G_i. Finally let W be a weak* limit of $W^{(n)}$ (we consider M_p as the dual space of A_p [3]). Then $W(\gamma) = W^{(n)}(\gamma)$ if $\gamma \in \Gamma_1 \times \cdots \times \Gamma_n$. We must show that W vanishes at infinity and is not the limit in the norm of M_p of finitely supported multipliers. Let $\epsilon > 0$ be given and let $1/n < \epsilon$. Denote by K_n the finite set $K_n = \{\gamma_1 \times \cdots \times \gamma_n\colon \gamma_i \in \operatorname{supp} W_i' \subseteq \Gamma_i\}$. Let $\gamma \notin K_n$ and suppose $W(\gamma) \neq 0$.

Let $\gamma = \gamma_1 \times \cdots \times \gamma_m$; with $\gamma_i \in \Gamma_i$ and $\gamma_m \neq \gamma_0$. Since $\gamma \notin K_n$ and $W(\gamma) \neq 0$, then $m > n$. Now $W(\gamma) = W_1'(\gamma_1) \otimes \cdots \otimes W_m'(\gamma_m)$. Therefore $\|W(\gamma)\| \leq \|W_1'(\gamma_1)\| \cdots \|W_m'(\gamma_m)\| \leq \|W_m'(\gamma)\|$, but since $\gamma_m \neq \gamma_0$, $W_m'(\gamma_m) = W_m(\gamma_m)$; therefore,

$$\|W(\gamma)\| \leq \|W_m(\gamma_m)\| \leq \sup_{\gamma \in \Gamma_m} \|W_m(\gamma)\| = \|W_m\|_{M_2} < 1/n.$$

We have proved that if $\gamma \in K_n$, $\|W(\gamma)\| < 1/n$ and hence W vanishes at infinity. On the other hand W cannot be the limit in the norm of M_p of finitely supported multipliers, for suppose $\widetilde{W}$ is a finitely supported multiplier satisfying $\|\widetilde{W} - W\|_{M_p} < \alpha(p)/4$. Since the support of $\widetilde{W}$ is finite, for some m, $\Gamma_m \cap \operatorname{supp} \widetilde{W} \subseteq \{\gamma_0\}$.

Now $\|W_m\|_{M_p} > \alpha(p)/2$ so there exists a trigonometric polynomial f with $\|f\|_p = 1$, $\operatorname{supp} \hat{f} \subseteq \Gamma_m$ such that $\|T_{W_m} f\|_p > \alpha(p)/2$.

Let $E \in \mathfrak{S}(\Gamma)$ denote the characteristic function of γ_0. Then $E \in M_p(G)$, $\|E\|_{M_p} = 1$, and $E + W_m = W_m'$. Also $W|\Gamma_m = W_m'$ and $W(\gamma_0) = I$. Thus one verifies that

$$(T_W - T_{\widetilde{W}})f = T_E(T_W - T_{\widetilde{W}})f + T_{W_m} f.$$

Therefore

$$\begin{aligned} \frac{\alpha(p)}{4} > \|W - \widetilde{W}\|_{M_p} &\geq \|(T_W - T_{\widetilde{W}})f\|_p \\ &\geq \|T_{W_m} f\|_p - \|T_E(T_W - T_{\widetilde{W}})f\|_p \\ &> \alpha(p)/2 - \alpha(p)/4 = \alpha(p)/4, \end{aligned}$$

a contradiction. This completes the proof of the Theorem.

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