

## EXPONENTIAL ESTIMATES FOR SOLUTIONS OF $y'' - q^2y = 0$

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**ABSTRACT.** It is shown for any nonnegative continuous function  $q$  on  $[0, \infty)$  and any  $c < 1$  that any positive increasing solution  $y$  of  $y'' - q^2y = 0$  satisfies  $y(x) \geq y(0) \exp(c \int_0^x q(t) dt)$  on the complement of a set of finite Lebesgue measure. It is also shown that if  $\liminf(\int_0^x q(t) dt/x) > 0$  then the equation has an exponentially increasing solution and an exponentially decreasing solution.

1. **Introduction.** We shall be concerned with the behavior of solutions of

$$(1) \quad y'' - q^2y = 0$$

on  $[0, \infty)$ . We prove first for any nonnegative continuous function  $q$  and any  $c < 1$  that any positive increasing solution  $y$  of (1) satisfies  $y(x) \geq y(0) \exp(c \int_0^x q(t) dt)$  for all  $x$  in the complement of a set of finite Lebesgue measure. We give an example to show that for some  $q$  this exceptional set is necessary and, on the other hand, show that if  $q'/q^2$  is bounded then the estimate holds for all  $x$  provided  $c$  is sufficiently small. Asymptotic estimates of this form have been studied by several authors (see for instance Bellman [1], Hartman and Wintner [2], and Hille [3, §§9.2 and 9.4]), but these estimates, although much more precise, depend on fairly strong assumptions about  $q^2$ .

In the third section we consider functions  $q$  for which  $\lim_{x \rightarrow \infty} \inf(\int_0^x q(t) dt/x)$  is positive, and we show that for such  $q$ , (1) has an exponentially increasing solution and an exponentially decreasing solution. This strengthens a result of Putnam [5] who obtained such solutions when  $\lim_{x \rightarrow \infty} \inf q$  is positive. Our conclusion is reminiscent of general results of Massera and Schäffer [4] on exponential dichotomies. For an exponential dichotomy these authors require in addition that the derivatives of the solutions of (1) also increase or decrease exponentially, and that the angular distance

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between the increasing and decreasing solutions remain bounded away from zero. They show that this occurs when a certain differential operator has closed range. However a bounded function  $q$  may have positive mean value, and thus, by our result, exponentially increasing and decreasing solutions, even though it vanishes on a sequence of intervals whose lengths increase without bound. For such  $q$  it is not difficult to see that the appropriate differential operator cannot have closed range (consider a suitable sequence of functions with compact support disjoint from the support of  $q$ ), and it then in fact follows from the results of Massera and Schäffer that an exponential dichotomy as outlined above cannot occur.

In proving the results mentioned above we find it convenient to work with the Riccati equation associated with (1) in the usual way. The results established in this fashion for Riccati equations are stated separately as Corollaries 2.2 and 3.2.

**2. Increasing solutions.** Our general estimate is the following.

**Theorem 2.1.** *Let  $q$  be a nonnegative continuous function on  $[0, \infty)$ . Let  $y$  be any positive solution of (1) such that  $y'(0) > 0$ . Then for every  $c, 0 < c < 1$ , there is a subset  $E_c$  of  $[0, \infty)$  such that*

$$m(E_c) \leq \frac{y(0)c^2}{y'(0)(1-c^2)} \quad \text{and} \quad y(x) \geq y(0) \exp\left(c \int_0^x q(t) dt\right)$$

on  $[0, \infty) \setminus E_c$ .

**Proof.**  $w = y'/y$  is positive on  $[0, \infty)$  and satisfies the Riccati equation

$$(2) \quad w' + w^2 = q^2.$$

Note that  $y(x) = y(0) \exp \int_0^x w(t) dt$ . Thus we must show

$$(3) \quad \int_0^x w(t) dt \geq c \int_0^x q(t) dt$$

on  $[0, \infty) \setminus E_c$ . From (2) we have  $w'/w + w = q^2/w$  or, by an integration,

$$\int_0^x q^2(t)/w(t) dt = \int_0^x w(t) dt + \log w(x) - \log w(0).$$

Using this and the Schwarz inequality we obtain

$$\begin{aligned} \left[ \int_0^x q(t) dt \right]^2 &\leq \int_0^x w(t) dt \int_0^x q^2(t)/w(t) dt \\ &= \int_0^x w(t) dt \left[ \int_0^x w(t) dt + \log w(x) - \log w(0) \right]. \end{aligned}$$

This inequality implies (3) for any value of  $x$  for which  $\log w(x) - \log w(0) \leq [(1 - c^2)/c^2] \int_0^x w(t) dt$  or, equivalently,  $w(x) \leq w(0) \exp(a \int_0^x w(t) dt)$  where we have set  $a = (1 - c^2)/c^2$ . Denote by  $E_c$  the set of all  $x$  for which this inequality does not hold. Now

$$\int_0^\infty w(x) \exp\left(-a \int_0^x w(t) dt\right) dx = \left(\frac{-1}{a}\right) \exp\left(-a \int_0^x w(t) dt\right) \Big|_0^\infty \leq \frac{1}{a}.$$

Hence, since  $w > 0$ ,

$$\frac{1}{a} \geq \int_{E_c} w(x) \exp\left(-a \int_0^x w(t) dt\right) dx \geq w(0)m(E_c).$$

It follows that

$$m(E_c) \leq 1/aw(0) = y(0)c^2/y'(0)(1 - c^2)$$

and that (3) holds on the complement of this set.

We record next the result about the Riccati equation (2) established above.

**Corollary 2.2.** *Let  $w$  be any positive solution of (2) on  $[0, \infty)$ . Then for any  $c$ ,  $0 < c < 1$ , there is a subset  $E_c$  of  $[0, \infty)$  such that  $m(E_c) \leq c^2/w(0)(1 - c^2)$  and  $\int_0^x w(t) dt \geq c \int_0^x q(t) dt$  for all  $x \in [0, \infty)/E_c$ .*

In our discussion of exponentially increasing solutions in the next section we shall find the following version of Theorem 2.1 useful.

**Corollary 2.3.** *Let  $y$  be a positive solution of (1) such that  $y'(0) > 0$ . For  $0 < c < 1$  set  $K(c) = y(0)c^2/y'(0)(1 - c^2)$ . Then for all  $x \geq K(c)$ ,  $y(x) \geq \exp(c \int_0^x -K(c) q(t) dt)$ .*

**Proof.** By Theorem 2.1 there is  $X \in [x - K(c), x]$  such that  $y(X) \geq \exp(c \int_0^X q(t) dt)$ . Thus

$$y(x) \geq y(X) \geq \exp\left(c \int_0^X q(t) dt\right) \geq \exp\left(c \int_0^x -K(c) q(t) dt\right).$$

We next show that the exceptional sets  $E_c$  of Theorem 2.1 cannot in general be avoided even for  $c$  very close to zero. We shall construct a function  $q$  such that for each increasing solution  $y$  of (1) and each  $c > 0$  there is a sequence of intervals on which  $y(x) < y(0) \exp(c \int_0^x q(t) dt)$ . To do this it suffices to show that if  $q$  has already been defined on  $[0, A]$ , then it can be extended to  $[A, B]$ ,  $B \geq N$ , so that on some subinterval  $[C, B]$ ,  $y(x) < \exp(2^{-N} \int_0^x q(t) dt)$  whenever  $y(0) = 1$  and  $y'(0) \leq N$ .

Set  $D = \max(A, N)$ . If  $D > A$ ,  $q$  may be chosen to be any nonnegative continuous function on  $[A, D]$ . Now define  $q(x) = (K - 2^N x)^{-1}$  on  $[D, B]$  where  $B$  has yet to be chosen and  $K = 2^N D + 1/q(D)$ . We note in passing that  $q$  satisfies  $q' = 2^N q^2$  on  $[D, B]$ . The solutions of (1) on  $[D, B]$  are of the form  $c_1(K - 2^N x)^{r_1} + c_2(K - 2^N x)^{r_2}$  where  $r_1 = \frac{1}{2} + 2^{-N-1}(2^{2N} + 4)^{\frac{1}{2}}$  and  $r_2 = \frac{1}{2} - 2^{-N-1}(2^{2N} + 4)^{\frac{1}{2}}$ . Also

$$\exp\left(2^{-N} \int_D^x q(t) dt\right) = [q(D)(K - 2^N x)]^{r_3}$$

where  $r_3 = -2^{-2N}$  so that for some  $M > 0$ ,

$$\exp\left(2^{-N} \int_0^x q(t) dt\right) = M(K - 2^N x)^{r_3}.$$

Since  $r_1 > 0$  and  $r_2 > r_3$ , there exists  $C > D$  such that the solution  $y$  of (1) with  $y(0) = 1$  and  $y'(0) = N$  satisfies  $y(C) < \exp(2^{-N} \int_0^C q(t) dt)$ . Thus if we choose  $B \in (C, D + 1/2^N q(D))$ , then  $y(x) < \exp(2^{-N} \int_0^x q(t) dt)$  on  $[C, B]$  whenever  $y(0) = 1$  and  $y'(0) \leq N$ .

The function  $q$  just constructed satisfies  $q' = 2^N q^2$  on  $[D, B]$ . We shall now see that if  $q'/q^2$  is bounded, then  $y(x) \geq \exp(c \int_0^x q(t) dt)$  for all  $x$  and all sufficiently small  $c$ . This will be a simple consequence of the following comparison theorem.

**Theorem 2.4.** *Let  $p_0$  and  $p_1$  be continuous functions on  $[0, \infty)$  with  $p_1 \geq p_0$ . If  $y_0$  is a positive solution of  $y'' - p_0 y = 0$ , then every solution  $y$  of  $y'' - p_1 y = 0$  with  $y(0) \geq y_0(0)$ , and  $y'(0)/y(0) \geq y'_0(0)/y_0(0)$  satisfies  $y \geq y_0$  and  $y' \geq y'_0$  on  $[0, \infty)$ .*

**Proof.** By the Sturm comparison theorem,  $y$  has at most one zero on  $[0, \infty)$ . We shall first show that in fact  $y$  has no zeros. Let  $y(x_0) = 0$ . Set  $w_0 = y'_0/y_0$  and  $w = y'/y$ . Then  $w(0) \geq w_0(0)$ ,  $w'_0 = p_0 - w_0^2$  on  $[0, \infty)$  and  $w' = p_1 - w^2 \geq p_0 - w^2$  on  $[0, x_0)$ . Hence, by the theory of differential inequalities,  $w \geq w_0$  on  $[0, x_0)$ . On the other hand, since  $\log y(x) \rightarrow -\infty$  as  $x \rightarrow x_0$  and  $\log y(x) = \log y(0) + \int_0^x w(t) dt$ , there is an increasing sequence  $\{x_n\}_{n=1}^\infty$  such that  $x_n \rightarrow x_0$  and  $w(x_n) \rightarrow -\infty$ . From this contradiction it follows that  $y$  has no zeros and thus that  $w = y'/y$  is defined on  $[0, \infty)$  and satisfies  $w' \geq p_0 - w^2$  there. Hence  $y'/y = w \geq w_0 = y'_0/y_0$  on  $[0, \infty)$ . But then  $\log y(x) - \log y(0) \geq \log y_0(x) - \log y_0(0)$  which gives  $y \geq y_0$  and then  $y' \geq y'_0$  on  $[0, \infty)$ .

From this we can easily prove

**Theorem 2.5.** *If  $q' \leq Mq^2$  on  $[0, \infty)$ , then for any  $c \leq (-M + (M^2 + 4)^{1/2})/2$  we have that  $y(x) \geq y(0)\exp(c \int_0^x q(t)dt)$  for all  $x$  whenever  $y$  is a positive solution of (1) such that  $y'(0)/y(0) \geq cq(0)$ .*

**Proof.**  $y(0)\exp(c \int_0^x q(t)dt)$  satisfies  $y'' - p_0y = 0$  with  $p_0 = cq' + c^2q^2 \leq (cM + c^2)q^2 \leq q^2$ . Thus the result follows immediately from Theorem 2.4.

**3. Exponential solutions.** We now introduce the assumption  $\lim_{x \rightarrow \infty} \inf \int_0^x q(t)dt/x > 0$ . Our result is the following.

**Theorem 3.1.** *Suppose  $\lim_{x \rightarrow \infty} \inf \int_0^x q(t)dt/x > M > 0$ . Then there are solutions  $y$  and  $z$  of (1) such that for any  $a > 0$  there exists  $b \geq a$  so that  $y(x) \geq y(a) \exp M(x - a)$ , and  $0 < z(x) \leq z(a) \exp M(a - x)$  for all  $x \geq b$ .*

**Proof.** It will suffice to consider the case  $a = 0$ , for it will be clear from the proof that we may then apply this case to the restriction of (1) to  $[a, \infty)$ .

Let  $y$  be any solution of (1) with  $y(0) > 0$  and  $y'(0) > 0$ . Choose  $M_1$  and  $c$  so that  $M < M_1 < \lim \inf(\int_0^x q(t)dt/x)$ ,  $0 < c < 1$ , and  $M < cM_1$ . Also choose  $x_1$  so that  $\int_0^x q(t)dt \geq M_1x$  for all  $x \geq x_1$ . Then, by Corollary 2.3,

$$y(x) \geq \exp\left(c \int_0^{x-K(c)} q(t) dt\right) \geq \exp cM_1(x - K(c))$$

for  $x \geq x_1 + K(c)$ . Hence  $y(x) \geq \exp Mx$  for all sufficiently large  $x$ .

It is well known that (1) has a unique positive bounded solution  $z$  such that  $z(0) = 1$ . Set  $w = z'/z$ . Then  $w$  is the unique negative solution of (2) on  $[0, \infty)$ , and  $z(x) = \exp \int_0^x w(t)dt$ . Thus in order to complete the proof it suffices to show  $\lim \inf(-\int_0^x w(t)dt/x) \geq M_1$ . We may assume that  $-w(0) \leq M_1$ , for our assertion certainly holds if  $-w(x) > M_1$  for all  $x$ , and we may otherwise restrict our attention to the interval  $[m, \infty)$  where  $m = \min\{x : -w(x) \leq M_1\}$ .

We shall show first that there exist arbitrarily large values of  $x$  for which  $-w(x) > M_1$ . For if  $w(x) \geq -M_1$  for all  $x \geq x_0$ , then integrating (2) and using the fact that  $w$  remains negative yields that for all  $x \geq x_0$ ,

$$M_1 \geq \int_{x_0}^x w'(t) dt = \int_{x_0}^x q^2(t) dt - \int_{x_0}^x w^2(t) dt \geq \int_{x_0}^x q^2(t) dt - M_1^2(x - x_0).$$

Hence,

$$M_1(x - x_0) + M_1^2(x - x_0)^2 \geq (x - x_0) \int_{x_0}^x q^2(t) dt \geq \left[ \int_{x_0}^x q(t) dt \right]^2$$

from the Schwarz inequality. But then  $M_1^2 + M_1/(x - x_0) \geq [\int_{x_0}^x q(t)dt/(x - x_0)]^2$ , which contradicts the assumption that  $\liminf(\int_0^x q(t)dt/x) > M_1$ . Thus there are arbitrarily large values of  $x$  for which  $-w(x) > M_1$ .

Now, as in the proof of Theorem 2.1, we have from (2) that

$$\int_0^x q^2(t)/-w(t) dt = \int_0^x -w(t) dt + \log |w(0)| - \log |w(x)|.$$

Hence,

$$\begin{aligned} \left[ \int_0^x q(t) dt \right]^2 &\leq \int_0^x -w(t) dt \int_0^x q^2(t)/-w(t) dt \\ &= \int_0^x -w(t) dt \left[ \int_0^x -w(t) dt + \log |w(0)| - \log |w(x)| \right]. \end{aligned}$$

Recall that for  $x \geq x_1$ ,  $\int_0^x q(t)dt \geq M_1x$ . If  $\int_0^{x_2} -w(t)dt < M_1x_2$  for some  $x_2 > x_1$ , then from the above inequality  $-w(x_2) < -w(0) \leq M_1$ . Let  $x_3 = \min\{x > x_2 : -w(x) \geq M_1\}$ . Then  $\log |w(x_3)| \geq \log |w(0)|$  and thus  $\int_0^{x_3} -w(t)dt \geq \int_0^{x_3} q(t)dt \geq M_1x_3$ . On the other hand,

$$\int_0^{x_3} -w(t) dt = \int_0^{x_2} -w(t) dt + \int_{x_2}^{x_3} -w(t) dt < M_1x_2 + M_1(x_3 - x_2) = M_1x_3.$$

Thus we must have  $\int_0^x -w(t)dt \geq M_1x$  for all  $x \geq x_1$ , and the theorem is established.

We have actually established the following property of the Riccati equation (2).

**Corollary 3.2.** *For any nonnegative continuous function  $q$ , the unique negative solution  $w$  of (2) satisfies  $\liminf(\int_0^x -w(t)dt/x) \geq \liminf(\int_0^x q(t)dt/x)$ , and any eventually positive solution  $w_0$  of (2) satisfies  $\liminf(\int_0^x w_0(t)dt/x) \geq \liminf(\int_0^x q(t)dt/x)$ .*

We close with the remark that the inequality just established for the negative solution  $w$  of (2) cannot in general be strengthened to one analogous to Corollary 2.2. For if  $q(x) = \sqrt{2/(1+x)}$ , then  $w(x) = -1/(1+x)$  and  $\int_0^x -w(t)dt \geq c \int_0^x q(t)dt$  cannot hold for  $x > 1/\sqrt{2}$ .

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