THE HARDY CLASS OF A FUNCTION
WITH SLOWLY-GROWING AREA

LOWELL J. HANSEN

ABSTRACT. In this paper we show that if \( f \) is analytic on the open unit disk and if the area of \( \{|z| \leq R\} \cap \text{image of } f \) grows sufficiently slowly as a function of \( R \), then \( f \) belongs to the Hardy class \( H^p \) for all \( p \) satisfying \( 0 < p < +\infty \).

Let \( \Delta \) denote the open unit disk \( \{|z| < 1\} \). It has been shown recently by H. Alexander, B. A. Taylor and J. L. Ullman [1, Theorem 1, p. 335] that

1. if \( f \) is analytic on \( \Delta \) and the area of \( f(\Delta) \) is finite, then \( f \) belongs to the Hardy class \( H^2 \); and

2. if in addition \( f(0) = 0 \), then \( \frac{1}{2} \|f\|_2^2 \leq \text{Area of } f(\Delta) \).

In this note we prove a theorem which strengthens (1) above:

Theorem. Let \( f \) be analytic on \( \Delta \). For \( R > 0 \), let

\[
A(R) = \text{Area of } \{|z| \leq R\} \cap f(\Delta).
\]

If \( A(R)(R^{-2} \log R) \to 0 \) as \( R \to +\infty \), then \( f \in H^p \) for all \( p \) satisfying \( 0 < p < +\infty \).

We remark that while weakening the hypothesis of (1) by replacing the condition "\( A(R) \) bounded" by the condition "\( A(R)(R^{-2} \log R) \to 0 \) as \( R \to +\infty \)" we have been able to strengthen the conclusion and get \( f \in H^p \) for all \( 0 < p < +\infty \). This new hypothesis is almost best possible since the inequality \( A(R)R^{-2} \leq \pi \) holds for any complex-valued function \( f \).

We remark further that the proof will make no use of the fact that the domain of \( f \) is the unit disk, and hence the Theorem remains valid if \( \Delta \) is replaced by any region.

Proof. Without loss of generality, we assume that \( f(0) = 0 \) and that \( f \) is unbounded. For \( t > 0 \), we define

\[
\theta(t) = \text{Angular Lebesgue measure of } \{|z| = t\} \cap f(\Delta),
\]

\[
\alpha(t) = \text{Angular Lebesgue measure of longest subarc of } \{|z| = t\} \cap f(\Delta),
\]

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and
\[ \chi(t) = 0 \quad \text{if } \{ |z| = t \} \subset \partial \Delta, \]
\[ = 1 \quad \text{if } \{ |z| = t \} \not\subset \partial \Delta. \]

For \( R > 1 \), let
\[ B(R) = \frac{\pi}{\log R} \int_1^R \frac{\chi(t)}{t \alpha(t)} \, dt. \]

Theorem 3.1 of [2] states that \( f \in H^p \) for all \( p \) satisfying \( 0 < p < \lim \inf_{R \to +\infty} B(R) \), and thus our theorem will follow if we can show that \( B(R) \to +\infty \) as \( R \to +\infty \). From the definitions of \( \theta \) and \( \alpha \) we get the inequalities
\[ A(R) = \int_0^R t \theta(t) \, dt \geq \int_0^R t \alpha(t) \, dt \]
\[ \geq \int_{[0,R] \cap \{ \alpha(t) = 2\pi \}} t \alpha(t) \, dt = 2\pi \int_{[0,R] \cap \{ \alpha(t) = 2\pi \}} t \, dt. \]

Let \( m(R) \) = Lebesgue measure of \( [0, R] \cap \{ \alpha(t) = 2\pi \} \). Then, since \( g(t) = t \) is an increasing function of \( t \), we get
\[ A(R) \geq 2\pi \int_0^R t \, dt = \pi [m(R)]^2. \]

Multiplying both sides of this inequality by \( R^{-2} \log R \) and recalling the original assumption on \( A(R) \), we conclude that \( R^{-1} m(R) \to 0 \) as \( R \to +\infty \).

Now \( R - 1 - m(R) \leq \int_0^R \chi(t) \, dt \), and hence
\[ (R - 1 - m(R))^2 \leq \left( \int_1^R \chi(t) \, dt \right)^2 \]
if \( R - 1 - m(R) \geq 0 \). An application of the Schwarz inequality yields
\[ \left( \int_1^R \chi(t) \, dt \right)^2 \leq \left( \int_1^R t \alpha(t) \, dt \right) \left( \int_1^R \frac{\chi(t)}{t \alpha(t)} \, dt \right) \leq A(R)(1/\pi)B(R) \log R. \]

Since \( R - 1 - m(R) \geq 0 \) for large \( R \), we get the inequality
\[ \pi [R - 1 - m(R)]^2 / A(R) \log R \leq B(R). \]

Using the fact that \( R^{-1} m(R) \to 0 \) and \( A(R)R^{-2} \log R \to 0 \) as \( R \to +\infty \), we conclude that \( B(R) \to +\infty \) as \( R \to +\infty \), which completes the proof.

REFERENCES
