PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

ON DILUTION AND CESÀRO SUMMATION

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ABSTRACT. The problem whether a real sequence (s_i) has a dilution which is $(C, 1)$ summable to a number s is transformed by means of two sequences measuring the oscillation of (s_i) about s. (If it does not oscillate, the condition, known, is that s is a limit point of (s_i).) For the jth consecutive block of s_i on one side of s, α_j is the minimum of their distances from s, β_j the sum of distances. Then there must exist positive numbers p_j such that $\beta_j + p_j\alpha_j = o(p_1 + \cdots + p_{j-1})$. The necessary condition and the sufficient condition coincide for very smooth sequences at $\alpha_i \log \beta_i = o(i)$.

Introduction. Diluting a series (usually in preparation for applying some summability method [3]) is inserting some zeros between its terms. Correspondingly, diluting a sequence is repeating consecutively some of its terms. This paper contributes to the problem: when does a sequence (s_i) have a dilution that is Cesàro summable to a given number s?

For bounded sequences, the problem is solved. Following V. Drobot [1], we note that the desired dilution does exist if s is a limit point of (s_i), and it does not exist if s is a nonlimit point and $\operatorname{sgn}(s_i - s)$ is finally constant. So we may assume that (s_i) breaks into consecutive finite blocks B_j in which $\operatorname{sgn}(s_i - s)$ is constant. For bounded s_i, the dilution exists if and only if the logarithm of the length of B_j is $o(j)$.

For general (s_i) (and suitable s as above) we define α_j as the minimum of $|s_i - s|$ over B_j, β_j as its sum over B_j. The problem is equivalent to the problem of finding positive numbers p_i such that $\beta_k + p_k\alpha_k = o(\sum_{i=1}^{i=k-1} p_i)$. Here $\beta_i \geq \alpha_i > \epsilon > 0$, with finitely many exceptions.

It is necessary for the dilution that $\log \beta_i = o(i)$. It is not necessary that $\alpha_i = o(i)$, but this must be true on some subsequence; equivalently, it is impossible that $i = O(\alpha_i)$. For sufficiently well-behaved sequences, it is necessary and sufficient that $\alpha_i \log \beta_i = o(i)$. The sufficiency proof assumes that the (necessary) approach of $i^{-1} \log \beta_i$ to zero is monotonic. Necessity

Received by the editors March 28, 1973.
AMS (MOS) subject classifications (1970). Primary 40G05.

is proved if the β_i are increasing and the ratios β_i/β_{i-1} decreasing and $i^{-1}\alpha_i \log \beta_i$ is smooth; more precisely, the first two hypotheses yield the necessary condition $i \neq O(\alpha_i \log \beta_i)$.

The referee points out previous work neighboring, but not interacting with, this. It can be found in and through [2].

Proofs. (s_i) will be a sequence of real numbers, (s_i') or (s_i^*) dilutions of it (as defined above and in [3]). Until stated otherwise, (s_i) is fixed and s is a fixed real number. By [1] we may assume, and do, that s is a nonlimit point of (s_i) and $\operatorname{sgn}(s_i - s)$ changes infinitely often; let us say (s_i) *surrounds* s. (This is an alternative to the popular practice of publishing theorems which are not true but become true after one has found all the "standing assumptions".)

Let J be the set of indices j for which $\operatorname{sgn}(s_j - s) \neq \operatorname{sgn}(s_{j-1} - s)$. It is an infinite set $\{i_1, i_2, \cdots\}$. Put $B_j = \{k: i_j \leq k < i_{j+1}\}$, $\alpha_j = \min[|s_i - s|: i \in B_j]$, $\beta_j = \Sigma[|s_i - s|: i \in B_j]$.

Lemma 1. *(s_i) has a dilution $(C, 1)$ summable to surrounded s if and only if it has a dilution (s_i') for which $\beta_j' = o(i_j')$.*

Proof. Of the Cesàro means M_j^*, M_{j+1}^* of any sequence (s_i^*) at i_j^* and i_{j+1}^*, write $M_j^* = s + \delta$; then $M_{j+1}^* - s = (i_j^*\delta + \beta_j^*)/i_{j+1}^*$. If (s_i^*) sums to s, $\beta_j^* = o(i_{j+1}^*)$. Since s is not a limit point of (s_i) or its dilution (s_i^*), $i_{j+1}^* - i_j^* = O(\beta_j^*)$; $\beta_j^* = o(i_j^*)$.

Conversely, given $\beta_j' = o(i_j')$, consider j_1 after which $\beta_j' < i_j'/2$. Either the Cesàro mean at the end of the next block in which $s_i > s$ exceeds s, or the mean at the end of the following block is less than s; in either case the next block B_j moves the mean toward s but not as much as ½ past it. By repeating B_j (i.e. every term in it) enough times, we can produce a mean M_j^*, of the dilution being constructed, lying between s and $s + \frac{1}{2}\operatorname{sgn}(s_{i_j}' - s)$. In this way we can keep the means M_i^* $(i \geq j)$ in $(s - 2^{-1}, s + 2^{-1})$, going back and forth across s so that all intermediate means are in this interval, until we reach j_2 after which $\beta_j' < i_j'/2^2$. Since $i_j^* \geq i_j'$, we can now bring and keep the means in $(s - 2^{-2}, s + 2^{-2})$; and so on.

Corollary. *For (s_i) to have a dilution $(C, 1)$ summable to surrounded s, it is necessary that $\beta_k^{1/k} \to 1$.*

Proof. Since s is not a limit point, $\liminf \beta_k^{1/k} \geq 1$. If the lim sup exceeds 1, this remains true for any dilution (which cannot decrease β_k, and preserves the block indices). Then for some $d > 1$ we have a sequence of values $\beta_k' > d^k$. Putting $r = (d-1)/2$, it is not finally true that $\beta_k' < r\Sigma_{i=1}^{i=k-1}\beta_i'$, for that would mean that finally, after $\beta_1' + \cdots + \beta_K' = C$, β_{K+j}' is at most $r(1+r)^j C$, and finally this is less than d^{K+j}. So $\beta_k' \neq o(R_k)$, $R_k = \Sigma_{i=1}^{i=k-1}\beta_i'$. But $i_k' = O(R_k)$ since s is not a limit point. Thus (s_i') does not satisfy the condition of the lemma.

Theorem 1. *The sequence* (s_i) *has a dilution* $(C, 1)$ *summable to the surrounded number* s *if and only if there exist positive numbers* p_i *such that*

$$\beta_k + p_k\alpha_k = o\left(\sum_{i=1}^{i=k-1} p_i\right).$$

Proof. If this condition holds, it is preserved by adding to each p_i any number between 0 and 1, for that increases the left side at most by $\alpha_k \leq \beta_k$. So we may choose the p_i integral. Then if we dilute by repeating p_k times the terms s_{j_k} which give the minima $\alpha_k = \min|s_j - s|$ in B_k, the lemma applies.

Conversely, given a dilution (s_i') with $\beta_j' = o(i_j')$, put $p_j = i_{j+1}' - i_j'$. Then i_k' is the right side of the displayed condition. The left side $\beta_k' + p_k\alpha_k' \leq 2\beta_k' = o(i_k')$, as required.

Theorem 2. (s_i) *has a dilution summable to surrounded* s *if* $i^{-1}\log\beta_i$ *decreases monotonically to* 0 *and* $\lim i^{-1}\alpha_i \log\beta_i = 0$.

Proof. The ith root of β_i, $1 + \epsilon_i$, decreases monotonically to 1, and ϵ_i to 0. For large i, $\log(1+\epsilon_i) \sim \epsilon_i$ and $\alpha_i = o(\epsilon_i^{-1})$. Putting $p_i = \beta_i$, $p_1 + \cdots + p_{k-1} = (1+\epsilon_1) + \cdots + (1+\epsilon_{k-1})^{k-1} \geq (1+\epsilon_k) + \cdots + (1+\epsilon_k)^{k-1} = p_k\epsilon_k^{-1} \gg p_k\alpha_k$ and $\gg \beta_k$.

Corollary. (s_i) *has a dilution summable to* s *if* α_i *is bounded and* $\log\beta_i = o(i)$.

Proof. Define $l_i = \sup[j^{-1}\log\beta_j \colon j \geq i]$; these decrease monotonically to 0. Define $\beta_i' = \exp il_i \geq \beta_i$. Since α_i is bounded, Theorem 2 shows that $p_i = \beta_i'$ solves the problem for (α_i) and (β_i'); hence it solves the problem for (α_i) and (β_i).

Lemma 2. *For any sequence of positive numbers* $p_0, p_1, \cdots,$

$$\log \sum_{i=0}^{i=n} p_i = O\left(\sum_{k=1}^{k=n} \left(p_k \Big/ \sum_{i=0}^{i=k-1} p_i\right)\right).$$

Proof. Let $f(n) = p_0 + \cdots + p_n$ and interpolate linearly. Then $\log f$ is everywhere continuous and differentiable except at integers. Its derivative in $(k-1, k)$ is less than $p_k/f(k-1)$; and $\log f(n) = O(\int_0^n d \log f(x))$.

Theorem 3. *(s_i) has no dilution summable to s if β_i is increasing and $\alpha_i \log(\beta_i/\beta_{i-1})$ is bounded away from zero.*

Proof. We have $\alpha_i^{-1} = O(\Delta(\log \beta_i))$. The required p's, in view of the lemma, would give

$$p_k \alpha_k = o(f(k-1)); \qquad p_k/f(k-1) = o(\alpha_k^{-1});$$

$\log \beta_k = o(\log f(k)) = o(\log \beta_k)$, which is absurd.

Corollary. *(s_i) has no dilution summable to s if β_i is increasing, $\log(\beta_i/\beta_{i-1})$ is decreasing, and $i = O(\alpha_i \log \beta_i)$.*

Proof. If the logarithm of the ratio, i.e. $\Delta(\log \beta_i)$, is decreasing, then $i\Delta(\log \beta_i) = O(\log \beta_i)$ and the theorem applies.

However the β's behave, one has the

Proposition. *(s_i) has no dilution summable to s if $i = O(\alpha_i)$.*

Proof. If $i = O(\alpha_i)$, the required p_i must be unbounded. In the subsequence of all j's for which $p_j = \max\{p_1, \cdots, p_j\}$, $p_j \alpha_j = o(p_1 + \cdots + p_{j-1}) = o(jp_j)$; $\alpha_j = o(j)$.

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