ON DILUTION AND CESÁRO SUMMATION

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ABSTRACT. The problem whether a real sequence \( (s_i) \) has a dilution which is \((C, 1)\) summable to a number \( s \) is transformed by means of two sequences measuring the oscillation of \( (s_i) \) about \( s \). (If it does not oscillate, the condition, known, is that \( s \) is a limit point of \( (s_i) \).) For the \( j \)th consecutive block of \( s_i \) on one side of \( s \), \( \alpha_j \) is the minimum of their distances from \( s \), \( \beta_j \) the sum of distances. Then there must exist positive numbers \( p_j \) such that \( \beta_j + p_j \alpha_j = o(p_1 + \cdots + p_{j-1}) \). The necessary condition and the sufficient condition coincide for very smooth sequences at \( \alpha_i \log \beta_i = o(i) \).

Introduction. Diluting a series (usually in preparation for applying some summability method [3]) is inserting some zeros between its terms. Correspondingly, diluting a sequence is repeating consecutively some of its terms. This paper contributes to the problem: when does a sequence \( (s_i) \) have a dilution that is Cesàro summable to a given number \( s \)?

For bounded sequences, the problem is solved. Following V. Drobot [1], we note that the desired dilution does exist if \( s \) is a limit point of \( (s_i) \), and it does not exist if \( s \) is a nonlimit point and \( sgn(s_i - s) \) is finally constant. So we may assume that \( (s_i) \) breaks into consecutive finite blocks \( B_j \) in which \( sgn(s_i - s) \) is constant. For bounded \( s_i \), the dilution exists if and only if the logarithm of the length of \( B_j \) is \( o(j) \).

For general \( (s_i) \) (and suitable \( s \) as above) we define \( \alpha_i \) as the minimum of \( |s_i - s| \) over \( B_j \), \( \beta_j \) as its sum over \( B_j \). The problem is equivalent to the problem of finding positive numbers \( p_i \) such that \( \beta_k + p_k \alpha_k = o(\sum_{i=1}^{k-1} p_i) \). Here \( \beta_i \geq \alpha_i > \epsilon > 0 \), with finitely many exceptions.

It is necessary for the dilution that \( \log \beta_i = o(i) \). It is not necessary that \( \alpha_i = o(i) \), but this must be true on some subsequence; equivalently, it is impossible that \( i = O(\alpha_i) \). For sufficiently well-behaved sequences, it is necessary and sufficient that \( \alpha_i \log \beta_i = o(i) \). The sufficiency proof assumes that the (necessary) approach of \( i^{-1} \log \beta_i \) to zero is monotonic. Necessity
is proved if the $\beta_i$ are increasing and the ratios $\beta_i/\beta_{i-1}$ decreasing and $i^{-1}a_i \log \beta_i$ is smooth; more precisely, the first two hypotheses yield the necessary condition $i \neq O(a_i \log \beta_i)$.

The referee points out previous work neighboring, but not interacting with, this. It can be found in and through [2].

Proofs. $(s_i)$ will be a sequence of real numbers, $(s_i')$ or $(s_i^*)$ dilutions of it (as defined above and in [3]). Until stated otherwise, $(s_i)$ is fixed and $s$ is a fixed real number. By [1] we may assume, and do, that $s$ is a nonlimit point of $(s_i)$ and $\text{sgn}(s_i - s)$ changes infinitely often; let us say $(s_i)$ sur-
rounds $s$. (This is an alternative to the popular practice of publishing theo-
rems which are not true but become true after one has found all the "standing assumptions").

Let $J$ be the set of indices $j$ for which $\text{sgn}(s_j - s) \neq \text{sgn}(s_{j-1} - s)$. It is an infinite set $\{i_1, i_2, \cdots \}$. Put $B_j = \{k: i_j \leq k < i_{j+1}\}$, $a_j = \min|s_i - s|: i \in B_j$, $\beta_j = \sum|s_i - s|: i \in B_j$.

Lemma 1. $(s_i)$ has a dilution $(C, 1)$ summable to surrounded $s$ if and only if it has a dilution $(s_i')$ for which $\beta_j = o(i_j')$.

Proof. Of the Cesàro means $M_j, M_{j+1}$ of any sequence $(s_i)$ at $i_j$ and $i_{j+1}$, write $M_j = s + \delta$; then $M_{j+1} - s = (i_{j+1}^* + \beta_{j+1}^*)/i_{j+1}^*$. If $(s_i^*)$ sums to $s$, $\beta_{j+1}^* = o(i_{j+1}^*)$. Since $s$ is not a limit point of $(s_i)$ or its dilution $(s_i^*)$, $i_{j+1}^* - i_j^* = O(\beta_j^*)$; $\beta_{j+1}^* = o(i_{j+1}^*)$.

Conversely, given $\beta_j = o(i_j')$, consider $i_j$ after which $\beta_j < i_j'/2$. Either the Cesàro mean at the end of the next block in which $s_i > s$ exceeds $s$, or the mean at the end of the following block is less than $s$; in either case the next block $B_j$ moves the mean toward $s$ but not as much as $\frac{1}{2}$ past it. By repeating $B_j$ (i.e. every term in it) enough times, we can produce a mean $M_j^*$, of the dilution being constructed, lying between $s$ and $s + \frac{1}{2} \text{sgn}(s_i' - s)$. In this way we can keep the means $M_j^*$ $(i \geq j)$ in $(s - 2^{-1}, s + 2^{-1})$, going back and forth across $s$ so that all intermediate means are in this interval, until we reach $j_2$ after which $\beta_j' < i_j'/2^2$. Since $i_j^* \geq i_j'$, we can now bring and keep the means in $(s - 2^{-2}, s + 2^{-2})$; and so on.

Corollary. For $(s_i)$ to have a dilution $(C, 1)$ summable to surrounded $s$, it is necessary that $\beta_{1/k} \rightarrow 1$.

Proof. Since $s$ is not a limit point, $\liminf \beta_{1/k} \geq 1$. If the lim sup
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exceeds 1, this remains true for any dilution (which cannot decrease $\beta_k$, and preserves the block indices). Then for some $d > 1$ we have a sequence of values $\beta_k' > d^k$. Putting $r = (d - 1)/2$, it is not finally true that $\beta_k' < r\sum_{i=1}^{i=k-1} \beta_k^i$, for that would mean that finally, after $\beta_1' + \cdots + \beta_k' = C$, $\beta_{K+j}^i$ is at most $r(1 + r)^jC$, and finally this is less than $d^{K+j}$. So $\beta_k' = o(R_k)$, $R_k = \sum_{i=1}^{i=k-1} \beta_i'$. But $i_k' = O(R_k)$ since $s$ is not a limit point. Thus $(s_i')$ does not satisfy the condition of the lemma.

**Theorem 1.** The sequence $(s_i)$ has a dilution $(C, 1)$ summable to the surrounded number $s$ if and only if there exist positive numbers $p_i$ such that

$$\beta_k + p_k \alpha_k = o \left( \sum_{i=1}^{i=k-1} p_i \right).$$

**Proof.** If this condition holds, it is preserved by adding to each $p_i$ any number between 0 and 1, for that increases the left side at most by $\alpha_k \leq \beta_k$. So we may choose the $p_i$ integral. Then if we dilute by repeating $p_k$ times the terms $s_{j_k}$ which give the minima $\alpha_k = \min|s_j - s|$ in $B_k$, the lemma applies.

Conversely, given a dilution $(s_i')$ with $\beta_j' = o(i_j')$, put $p_j = i_{j+1}' - i_j'$. Then $i_k'$ is the right side of the displayed condition. The left side $\beta_k + p_k \alpha_k < 2\beta_k = o(i_k')$, as required.

**Theorem 2.** $(s_i)$ has a dilution summable to surrounded $s$ if $i^{-1} \log \beta_i$ decreases monotonically to 0 and $\lim i^{-1} \alpha_i \log \beta_i = 0$.

**Proof.** The $i$th root of $\beta_i, 1 + \epsilon$, decreases monotonically to 1, and $\epsilon_i$ to 0. For large $i$, $\log(1 + \epsilon) \sim \epsilon$ and $\alpha_i = o(\epsilon_i^{-1})$. Putting $p_0 = \beta_i, p_1 + \cdots + p_{k-1} = (1 + \epsilon_1) + \cdots + (1 + \epsilon_{k-1})^{k-1} \geq (1 + \epsilon_k) + \cdots + (1 + \epsilon_k)^{k-1} = p_k \epsilon_k^{-1} \alpha_k$ and $\alpha_k \beta_k$.

**Corollary.** $(s_i)$ has a dilution summable to $s$ if $\alpha_i$ is bounded and $\log \beta_i = o(\alpha_i)$.

**Proof.** Define $l_i = \sup[j^{-1} \log \beta_j, j \geq i]$; these decrease monotonically to 0. Define $\beta_i' = \exp il_i \geq \beta_i$. Since $\alpha_i$ is bounded, Theorem 2 shows that $p_i = \beta_i'$ solves the problem for $(\alpha_i)$ and $(\beta_i)$; hence it solves the problem for $(\alpha_i)$ and $(\beta_i)$.

**Lemma 2.** For any sequence of positive numbers $p_0, p_1, \cdots$,
Proof. Let \( f(n) = p_0 + \cdots + p_n \) and interpolate linearly. Then \( \log f \) is everywhere continuous and differentiable except at integers. Its derivative in \((k-1, k)\) is less than \( p_k / f(k-1) \); and \( \log f(n) = O(\int_0^n d \log f(x)) \).

Theorem 3. \((s_i)\) has no dilution summable to \( s \) if \( \beta_i \) is increasing and \( \alpha_i \log(\beta_i / \beta_{i-1}) \) is bounded away from zero.

Proof. We have \( \alpha_i^{-1} = O(\Delta(\log \beta_i)) \). The required \( p_i \)'s, in view of the lemma, would give

\[ p_k \alpha_k = o(f(k-1)); \quad p_k / f(k-1) = o(\alpha_k^{-1}); \]

\[ \log \beta_k = o(\log f(k)) = o(\log \beta_k), \] which is absurd.

Corollary. \((s_i)\) has no dilution summable to \( s \) if \( \beta_i \) is increasing, \( \log(\beta_i / \beta_{i-1}) \) is decreasing, and \( i = o(\alpha_i \log \beta_i) \).

Proof. If the logarithm of the ratio, i.e., \( \Delta(\log \beta_i) \), is decreasing, then

\[ i \Delta(\log \beta_i) = O(\log \beta_i) \] and the theorem applies.

However the \( \beta_i \)'s behave, one has the

Proposition. \((s_i)\) has no dilution summable to \( s \) if \( i = O(\alpha_i) \).

Proof. If \( i = O(\alpha_i) \), the required \( p_i \) must be unbounded. In the subsequence of all \( j \)'s for which \( p_j = \max\{p_1, \cdots, p_j\} \), \( p_j \alpha_j = o(p_1 + \cdots + p_{j-1}) = o(jp_j) \); \( \alpha_j = o(j) \).

REFERENCES


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