RIESZ SEMINORMS WITH FATOU PROPERTIES

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ABSTRACT. A seminormed Riesz space \( L_\rho \) satisfies the \( \sigma \)-Fatou property (resp., the Fatou property) if \( \theta \leq u \downarrow u \) in \( L \) (resp., \( \theta \leq u \downarrow u \) in \( L \)) implies \( \rho(u_n) \downarrow \rho(u) \) (resp., \( \rho(u_a) \downarrow \rho(u) \)). The following results are proved:

(i) A normed Riesz space \( L_\rho \) satisfies the \( \sigma \)-Fatou property if, and only if, its norm completion does and \( L_\rho \) has \( (A, 0) \).

(ii) The quotient space \( L_\rho/I_\rho \) has the Fatou property if \( L_\rho \) is Archimedean with the Fatou property, \( (I_\rho = \{ u \in L: \rho(u) = 0 \}) \).

(iii) If \( L_\rho \) is almost \( \sigma \)-Dedekind complete with the \( \sigma \)-Fatou property, then \( L_\rho/I_\rho \) has the \( \sigma \)-Fatou property.

A counterexample shows that (iii) may be false for Archimedean Riesz spaces.

1. Riesz seminorms. For notation and terminology not explained below we refer the reader to [5]. A seminormed Riesz space \( L_\rho \) is a Riesz space \( L \) equipped with a seminorm \( \rho \) satisfying \( \rho(u) \leq \rho(v) \) whenever \( |u| \leq |v| \) holds in \( L \).

For seminormed Riesz spaces \( L_\rho \) the following properties were introduced:

\( (A, 0) \): \( u_n \downarrow \theta \) in \( L \) and \( \{ u_n \} \) is a \( \rho \)-Cauchy sequence implies \( \rho(u_n) \to 0 \).

\( (A, i) \): \( u_n \downarrow \theta \) in \( L \) implies \( \rho(u_n) \to 0 \).

\( (A, ii) \): \( u_a \downarrow \theta \) in \( L \) implies \( \rho(u_a) \to 0 \).

Following Luxemburg and Zaanen [4, Notes II and XIII] we also have:

Definition 1.1 (\( \sigma \)-Fatou property). A seminormed Riesz space \( L_\rho \) satisfies the \( \sigma \)-Fatou property whenever \( \theta \leq u_n \uparrow u \) in \( L \) implies \( \rho(u_n) \uparrow \rho(u) \).

(Fatou property). A seminormed Riesz space \( L_\rho \) satisfies the Fatou property whenever \( \theta \leq u_a \uparrow u \) in \( L \) implies \( \rho(u_a) \uparrow \rho(u) \).

Obviously the Fatou implies the \( \sigma \)-Fatou, \( (A, i) \) implies the \( \sigma \)-Fatou and \( (A, ii) \) implies the Fatou property. Also the \( \sigma \)-Fatou implies the \( (A, 0) \) property. Indeed, if \( \{ u_n \} \) is a \( \rho \)-Cauchy sequence with \( u_n \downarrow \theta \) in \( L \), then
\[ \theta \leq u_m - u_n \uparrow_{n \geq m} u_m \text{ in } L, \text{ for each fixed } m, \text{ and hence } \rho(u_m - u_n) \uparrow_{n \geq m} \rho(u_m). \text{ This implies } \rho(u_n) \to 0. \]

Example 1.2. (i) Let \( L \) be the Riesz space of all real sequences which are eventually constant. Let \( \rho(u) = |u(\infty)| + \sup \{|u_n| : n = 1, 2, \ldots \} \) for all \( u \in L \). \( u(\infty) = u(n) \) for all sufficiently large \( n \). Note that the \( \sigma \)-Fatou property does not hold in \( L_\rho \). However \( L_\rho \) does satisfy the \((A, 0)\) property.

(ii) Let \( L \) be as in (i) and let \( \rho(u) = \sup \{|u(n)| : n = 1, 2, \ldots \} \) for all \( u \). Then \( L_\rho \) is noncomplete with the Fatou property. Note that \((A, i)\) does not hold.

(iii) Let \( L \) be the Riesz space of all bounded real valued Lebesgue measurable functions defined on \([0, 1]\), with \( f \leq g \) if \( f(x) \leq g(x) \) for all \( x \in [0, 1] \). Let \( \rho(u) = \int_0^1 |u(x)| \, dx + \sup \{|u(x)| : x \in [0, 1] \} \) for all \( u \in L \). Note that \( L \) is \( \rho \)-complete with the \( \sigma \)-Fatou property but without the Fatou property.

(iv) The cartesian product of the spaces in (ii) and (iii) with the product norm gives a noncomplete normed Riesz space without the Fatou and \((A, i)\) properties, but with the \( \sigma \)-Fatou property. \( \square \)

We recall that a Riesz subspace \( L \) of a Riesz space \( M \) is said to be order dense in \( M \) if \( \sup \{v \in L : \theta \leq v \leq u\} = u \) holds in \( M \) for all \( u \in M^+ \). If \( M \) is Archimedean (and hence so is \( L \)) then the universal completion of \( M \) \([5, \text{pp. 338–341}]\) equally serves as the universal completion of \( L \); consequently \( M \) can be considered as a Riesz subspace of the universal completion of \( L \). Now, if \( L_\rho \) is a normed Riesz space with \((A, 0)\) then \( L_\rho \) is order dense in its norm completion \( \overline{L}_\rho \) \([3, \text{Theorem 61.5, p. 652}]\) and so \( \overline{L}_\rho \) "seats" in the universal completion of \( L \) as an order dense Riesz subspace. This observation will be used in the next theorem.

**Theorem 1.3.** If the normed Riesz space \( L_\rho \) satisfies the \( \sigma \)-Fatou property, then we have:

(i) The norm completion \( \overline{L}_\rho \) of \( L_\rho \) satisfies the \( \sigma \)-Fatou property.

(ii) \( \rho(u) = \inf \{ \lim \rho(u_n) : \{u_n\} \subseteq L^+, u_n \uparrow \text{ and } u_n \wedge |u| \uparrow |u| \text{ in } \overline{L}_\rho \} \), for every \( u \in \overline{L}_\rho \).

**Proof.** Let \( K \) be the universal completion of \( L \) \([5, \text{Theorem 50.8, p. 340}]\). Define \( \lambda \) on \( K \) by the formula:

\[ \lambda(u) = \inf \{ \lim \rho(u_n) : \{u_n\} \subseteq L^+; u_n \uparrow \text{ and } u_n \wedge |u| \uparrow |u| \text{ in } K \} \]

with \( \inf \emptyset = +\infty \). Then we have:

(i) \( \lambda(u) = \rho(u) \) for all \( u \) in \( L \).
To verify (i) use the $\sigma$-Fatou property of $\rho$.

(ii) $\lambda(u) = \lambda(|u|)$ for all $u$ in $K$, and $\theta \leq u \leq v$ in $K$ implies $\lambda(u) \leq \lambda(v)$.

(iii) $\lambda(u) \geq 0$ for all $u$ in $K$ and $\lambda(u) = 0$ implies $u = \theta$.

To see (iii) use the order density of $L$ in $K$.

(iv) $\lambda(u + v) \leq \lambda(u) + \lambda(v)$, $\lambda(\alpha u) = |\alpha| \lambda(u)$ for all $u, v$ in $K$ and all $\alpha$ in $R$.

(v) If $\{u_n\} \subseteq L^+$ and $\theta \leq u_n \uparrow u$ in $K$, then $\rho(u_n) \uparrow \lambda(u)$.

(vi) Let $U = \{u \in K^+: \theta \leq u_n \uparrow u$, for some sequence $\{u_n\} \subseteq L^+\}$. Assume $\theta \leq u_n \uparrow u$ in $K$, $\{u_n\} \subseteq U$ and $\lambda(u_n) \uparrow \alpha < +\infty$. Then $\theta \leq u_n \uparrow u$ in $K$ and $\lambda(u) = \alpha$ for some $u$ in $U$.

To see (vi) pick $\{u_{n,k}: k = 1, 2, \cdots \} \subseteq L^+$ such that $u_{n,k} \uparrow u_n (n = 1, 2, \cdots)$. Define $w_n = \sup\{u_{i,n}: i = 1, \cdots, n\} \in L^+$ ($n = 1, 2, \cdots$) and note that $\rho(w_n) \leq \alpha$ for all $n$. Now, let $\theta < v \in L$. Pick $m \in N$ such that $mp(v) = \rho(mv) > \alpha$ and observe that $w_n \wedge mv \uparrow mv$ implies $\rho(mv) \leq \alpha$. So, $\sup\{w_n \wedge mv: n = 1, 2, \cdots\} < mv$. This observation implies $\theta \leq w_n \uparrow u$ in $K$ [2, Proposition 1, p. 342]. (Since $E$ is order dense in $C_\infty(X)$, observe that Fremlin’s proof works if we replace the assumption “for every $x > 0$ in $C_\infty(X)$” by “for every $x > 0$ in $E$”.). Thus $\theta \leq w_n \uparrow u$ and $u \in U$. Now, combine (v) with the relation $w_n \leq u_n$ for all $n$ to obtain $\theta \leq u_n \uparrow u$ and $\lambda(u) = \alpha$.

(vii) Let $\theta \leq u$, $\lambda(u) < +\infty$ and let $\epsilon > 0$. Then there exists $\nu \in U$, $u \leq \nu$ such that $\lambda(\nu) \leq \lambda(u) + \epsilon$.

To verify (vii), pick $\{u_n\} \subseteq L^+$, $u_n \uparrow u$, $u_n \wedge |u| \uparrow |u|$ and such that $\lim \rho(u_n) \leq \lambda(u) + \epsilon$. As in case (vi) note that $u_n \uparrow \nu$ in $K$ for some $\nu$ of $U$. Now use (v) to obtain $\lambda(\nu) \leq \lambda(u) + \epsilon$.

(viii) Let $L_\lambda = \{u \in K: \lambda(u) < +\infty\}$. Then $L_\lambda$ is a complete normed Riesz space.

For (viii) use (vii) and a routine argument to show that $L_\lambda$ satisfies the Riesz-Fischer property and hence it is $\lambda$-complete [4, Theorem 26.3, Note VIII, p. 105].

(ix) The closure of $L_\rho$ in $L_\lambda$, $\overline{L_\rho}$, is the norm completion of $L_\rho$.

Now, let $\theta \leq u_n \uparrow u$ in $\overline{L_\rho}$. Since $L$ is order dense in $K$, $u_n \uparrow u$ also holds in $K$. Given $\epsilon > 0$, pick an element $u_0 \in \overline{L_\rho}$, $u \leq u_0$, $u_0 \in U$ with $\lambda(u_0 - u) < \epsilon$ (see [3, Theorem 60.3, p. 648]). Similarly pick $v_n \in \overline{L_\rho}$, $u_n \leq v_n \leq u_0$, $\lambda(v_n - u_n) \leq \epsilon/2^{n+1}$ and $v_n \in U$, $n = 1, 2, \cdots$. Put $w_n = \sup\{v_i: i = 1, \cdots, n\}$ ($n = 1, 2, \cdots$) and note $\lambda(w_n - u_n) \leq \epsilon$ and $u_n \leq w_n \leq u_0$ for all $n$. Hence $w_n \uparrow u_1 \leq u_0$ in $L_\lambda$ and so $u \leq u_1 \leq u_0$ in $L_\lambda$. But then $\lambda(u) \leq \lambda(u_1) = \lim \lambda(w_n) \leq \lim \lambda(u_n) + \epsilon$ for all $\epsilon > 0$. Hence $\lambda(u_n) \uparrow \lambda(u)$, i.e.,
\( \bar{L}_\rho \) satisfies the \( \sigma \)-Fatou property. Part (ii) follows immediately from the above construction. \( \square \)

**Corollary 1.4.** Let \( L_\rho \) be a normed Riesz space with norm completion \( \bar{L}_\rho \). Then the following statements are equivalent.

(i) \( L_\rho \) satisfies the \( \sigma \)-Fatou property.

(ii) \( L_\rho \) satisfies the \( \sigma \)-Fatou property and \( L_\rho \) has \( (A, 0) \).

**Proof.** To see that (ii) implies (i) use Theorem 61.5 of [3, p. 652]. \( \square \)

For \( L = C([0, 1]) \) and \( \rho(u) = \int_0^1 |u(x)| \, dx \) we have \( \bar{L}_\rho = L_1([0, 1]) \). Note that \( \bar{L}_\rho \) satisfies the \( \sigma \)-Fatou property (in fact the \( (A, ii) \) property). However, \( L_\rho \) does not satisfy the \( (A, 0) \) property [5, Exercise 18.14(i), p. 104].

We close this section recalling a notion useful for the next section. A Riesz space \( L \) is called almost \( \sigma \)-Dedekind complete if it can be embedded as a super order dense Riesz subspace of a \( \sigma \)-Dedekind complete Riesz space \( K \), i.e., if \( L \) is a Riesz subspace of \( K \) (more precisely \( L \) is Riesz isomorphic to a Riesz subspace of \( K \)) such that for every \( \theta \leq u \in K \), there exists a sequence \( \{u_n\} \subseteq L \) with \( \theta \leq u_n \uparrow u \) in \( K \) (see [1]).

2. The quotient Riesz space. \( L_\rho / I_\rho \). The null ideal of a given seminormed Riesz space \( L_\rho \) is denoted by \( I_\rho \), i.e., \( I_\rho = \{u \in L : \rho(u) = 0\} \). It is evident that \( I_\rho \) is a \( \sigma \)-ideal (resp. a band) if \( \rho \) satisfies the \( \sigma \)-Fatou property (resp. the Fatou property). It is also obvious that the quotient Riesz space \( L_\rho / I_\rho \) becomes a normed Riesz space under the norm \( [\rho]([u]) = \rho(u) \) (\( [u] \) denotes the equivalence class of \( u \)).

**Question:** If \( L_\rho \) satisfies the \( \sigma \)-Fatou property, does the normed Riesz space \( L_\rho / I_\rho \) satisfy the \( \sigma \)-Fatou property?

The next theorem gives a condition for the answer to be affirmative.

**Theorem 2.1.** Assume that the seminormed Riesz space \( L_\rho \) satisfies the \( \sigma \)-Fatou property and that \( L \) is almost \( \sigma \)-Dedekind complete. Then the normed Riesz space \( L_\rho / I_\rho \) satisfies the \( \sigma \)-Fatou property.

**Proof.** Let \( K \) be a \( \sigma \)-Dedekind complete Riesz space containing \( L \) as a super order dense Riesz subspace. We can assume that the ideal generated by \( L \) is all of \( K \). Given \( u \in K \) pick \( \{u_n\} \subseteq L \) with \( \theta \leq u_n \uparrow |u| \) in \( K \) and define \( \lambda(u) = \lim \rho(u_n) \). Note that \( \lambda(u) \) is independent of the sequence chosen and that \( \lambda \) is a Riesz seminorm of \( K \) with the \( \sigma \)-Fatou property and with \( \lambda = \rho \) on \( L \).

Let \( L / I_\lambda \) be the canonical image of \( L \) in \( K / I_\lambda \). Observe that \( L_\rho / I_\rho \)
is Riesz isomorphic to $L/I_\lambda$ (the mapping $[u] = u + I_\rho \rightarrow u + I_\lambda = [u]$ does it) and that the quotient norm $[\rho]$ on $L_\rho/I_\rho$ and the norm induced from $K_\lambda/I_\lambda$ to $L/I_\lambda$ coincide. Now let $[\theta] \leq [u_n] \uparrow [u]$ in $L_\rho/I_\rho$, so $[\theta] \leq [u_n] \uparrow [u]$ holds also in $L/I_\lambda$. We can assume $\theta \leq u_n \uparrow u$ in $L$, so $\theta \leq u_n \uparrow v \leq u$ holds in $K$ and hence $[\theta] \leq [u_n] \uparrow [v]$ in $K_\lambda/I_\lambda$ [5, Theorem 18.11, p. 103]. Since $L/I_\lambda$ is order dense in $K_\lambda/I_\lambda$, $[u_n] \uparrow [u]$ also holds in $K/I_\lambda$ and hence $[v] = [u]$, so $\lambda(v) = \lambda(u) = \rho(u)$.

Thus $[\rho]([u_n]) = \rho(u_n) = \lambda(u_n) \uparrow \lambda(v) = \rho(u) = [\rho]([u])$, and the proof is finished. □

**Question:** If we replace the almost $\sigma$-Dedekind completeness of $L$ by Archimedeanness is Theorem 2.1 still true?

The following example shows that the answer is negative in general.

**Example 2.2.** Let $L$ be the Archimedean Riesz space $C(R_\infty)$. ($R_\infty$ is the one point compactification of the real numbers considered with the discrete topology (see [5, Example (v), p. 141]). Note that $L$ is not almost $\sigma$-Dedekind complete. Now, define the Riesz seminorm $\rho$ on $L$, by $\rho(u) = |u(\infty)| + \sup \{|u(n)| : n = 1, 2, \ldots\}$. Note that $\rho$ satisfies the $\sigma$-Fatou property but not the Fatou property. (In fact $\rho$ satisfies the $(A, i)$ property.) Note also that $I_\rho$ is a band.

Now, let $u_n = \chi\{1, \ldots, n\}$, $n = 1, 2, \ldots$. Then $\theta \leq u_n \uparrow e$ in $L$ ($e(x) = 1$ for all $x \in R$) and $\rho(u_n) = 1$ for all $n$. It is easily seen that $[\theta] \leq [u_n] \uparrow [e]$ holds in $L_\rho/I_\rho$. But $[\rho]([u_n]) = \rho(u_n) = 1 \uparrow [\rho]([e]) = \rho(e) = 2$.

Hence $L_\rho/I_\rho$ does not satisfy the $\sigma$-Fatou property. □

A better situation holds if $\rho$ satisfies the Fatou property. The next theorem tells us that $L_\rho/I_\rho$ satisfies the Fatou property if $L_\rho$ does.

**Theorem 2.3.** Let $L_\rho$ be an Archimedean seminormed Riesz space with the Fatou property. Then the normed Riesz space $L_\rho/I_\rho$ satisfies the Fatou property.

**Proof.** Repeat the proof of Theorem 2.1 replacing $K$ by $L^\delta$, the Dedekind completion of $L$. □

**REFERENCES**


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