BOUNDDED SOLUTIONS OF THE EQUATION $\Delta u = pu$

ON A RIEMANNIAN MANIFOLD

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ABSTRACT. Given a nonnegative $C^1$-function $p(x)$ on a Riemannian manifold $R$, denote by $B_p(R)$ the Banach space of all bounded $C^2$-solutions of $\Delta u = pu$ with the sup-norm. The purpose of this paper is to give a unified treatment of $B_p(R)$ on the Wiener compactification for all densities $p(x)$. This approach not only generalizes classical results in the harmonic case ($p = 0$), but it also enables one, for example, to easily compare the Banach space structure of the spaces $B_p(R)$ for various densities $p(x)$. Typically, let $\beta(p)$ be the set of all $p$-potential nondensity points in the Wiener harmonic boundary $\Delta$, and $C_p(\Delta)$ the space of bounded continuous functions $f$ on $\Delta$ with $f|_{\Delta - \beta(p)} = 0$.

Theorem. The spaces $B_p(R)$ and $C_p(\Delta)$ are isometrically isomorphic with respect to the sup-norm.

Throughout this paper $R$ is an orientable Riemannian $C^\infty$-manifold of dim $\geq 2$, and $p(x)$ is a nonnegative $C^1$-function on $R$. Denote by $B_p(R)$ the space of bounded $C^2$-solutions $u$ on $R$ of the elliptic equation $\Delta u = pu$, where $\Delta u$ is the Laplacian of $u$ on $R$. As one studies bounded harmonic functions on the Wiener compactification, the space $B_p(R)$ has been investigated on the so-called Wiener $p$-compactification (cf. Loeb and Walsh [2], Wang [9]). However, their consideration restricts one to construct different compactifications for different densities $p(x)$.

The purpose of the present paper is to give a unified treatment of the spaces $B_p(R)$ on the Wiener compactification $R^*$ for all densities $p(x)$. This approach, for instance, enables one to easily compare the linear space structure of the spaces $B_p(R)$ for various densities $p(x)$. Typically, let $\beta(p)$ be the set of $p$-potential nondensity points $x$ in the Wiener harmonic boundary $\Delta$ (see below for its definition), and $C_p(\Delta)$ the space of bounded
continuous functions \( f \) on \( \Delta \) such that \( f|\Delta - \beta(p) = 0 \). Then \( B_p(R) \) and \( C_p(\Delta) \) are isometrically isomorphic with respect to the sup-norm.

For the notation and terminology we refer the reader to Sario and Nakai [8, Chapter 4].

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1. First we observe a simple fact.

**Lemma.** Every \( u \in B_p(R) \) is continuously extendable to the Wiener compactification \( R^* \) of \( R \). Furthermore \( u \) has the property \( \|u\| = \max_{\Delta} |u| \), where \( \|\cdot\| \) is the sup-norm and \( \Delta \) is the Wiener harmonic boundary.

A point \( x \in \Delta \) will be called a \( p \)-potential nondensity point if there exists an open neighborhood \( U^* \) of \( x \) in \( R^* \) such that

\[
\sup_{a \in U} \int_U G_U(a, y)p(y) \, dy < \infty,
\]

where \( U = U^* \cap R \), \( G_U(a, y) \) is the (harmonic) Green's function for \( U \), and \( dy \) is the (Riemannian) volume element of \( R \). Denote by \( \beta(p) \) the set of all \( p \)-potential nondensity points in \( \Delta \) (cf. Nakai [5]).

For \( p \not= 0 \) the above maximum principle is too crude for our purpose:

**Theorem.** Every \( u \in B_p(R) \) has the property \( \|u\| = \max_{\beta(p)} |u| \).

**Proof.** It suffices to show that \( u = 0 \) on \( \Delta - \beta(p) \). To the contrary suppose that \( u(x) = 2\epsilon > 0 \) for some \( x \in \Delta - \beta(p) \). Choose an open neighborhood \( U^* \) of \( x \) in \( R^* \) such that \( u > \epsilon \) on \( U^* \). Set \( U = U^* \cap R \). We may modify \( U \) to have a smooth \( \partial U \). Let \( \{\Omega_n\}_{n=1}^\infty \) be a "regular" exhaustion of \( U \). By Stokes' formula

\[
u(z) = h_n(z) - \int_{\Omega_n} G_n(z, y)p(y)u(y) \, dy
\]
on \( \Omega_n \), where \( h_n \in B_0(\Omega_n) \) with \( h_n|_{\partial \Omega_n} \equiv u \) and \( G_n(z, y) \) is the Green's function for \( \Omega_n \). Therefore it is seen that

\[0 \leq \int_{\Omega_n} G_n(z, y)p(y)u(y) \, dy \leq \|u\|\]
on \( \Omega_n \). By the monotone convergence theorem, we deduce that

\[
\int_U G_U(z, y)p(y) \, dy \leq \|u\|/\epsilon
\]
on $U$, a contradiction to the fact that $x \notin \beta(\rho)$.

2. For a parabolic $R$, the space $B_p(R) = \{0\}$ or the real number field according as $p \neq 0$ or $p = 0$ (Ozawa [6]). We thus assume that $R$ is hyperbolic. Set

$$H_pB(R) = \{u \in B_0(R)|u \equiv 0 \text{ on } \Delta - \beta(\rho)\}.$$

It is not difficult to see that $C_p(\Delta)$ and $H_pB(R)$ are isometrically isomorphic Banach spaces with the sup-norm $\|\cdot\|$.

**Theorem.** For any density $\rho(x)$ on $R$ the Banach spaces $B_p(R)$ and $C_p(\Delta)$ are isometrically isomorphic. In particular

$$C_p(\Delta) = \{u|\Delta: u \in B_p(R)\}.$$

**Proof.** It suffices to show that every $h \in C_p(\Delta)$ can be extended to a function in $B_p(R)$.

Without loss of generality we may assume that $h \in H_pB(R)$ and $h \geq 0$ on $R$. Define $\nu(z) = \sup \{|u(z)|u \in F_h\}$, where $F_h = \{u \in B_p(R)|0 \leq u \leq h \text{ on } R\}$. Since the class $F_h$ forms a Perron family for $\Delta u = pu$, it follows that $\nu \in B_p(R)$. We need to prove that $\nu \equiv h$ on $\beta(\rho)$.

On the contrary, assume that there exists a point $x \in \beta(\rho)$ such that $h(x) > \nu(x) \geq 0$. Let $\epsilon$ be a positive constant with $\nu(x) < \epsilon < h(x)$. Choose an open neighborhood $U^*$ of $x$ in $R^*$ such that $h > \epsilon > \nu$ on $U^*$, $U = U^* \cap R$ has smooth $\partial U$, and $\sup_{a \in U} \int_U G_U(a, y)p(y)dy < \infty$. Take $n$ so large that $\sup_{a \in U} \int_U G_U(a, y)p(y)dy < n$.

For any $\phi \in C(U)$, the space of bounded continuous functions on $U$, define an integral operator $T$ by

$$(T\phi)(z) = -\frac{1}{n} \int_U G_U(z, y)p(y)\phi(y)dy.$$

It is well known (cf. e.g. Miranda [3, p. 25]) that $T$ is a linear operator in $C(U)$ and its operator norm satisfies

$$\|T\| \leq \frac{1}{n} \sup_{a \in U} \int_U G_U(a, y)p(y)dy < 1.$$

Thus the Fredholm integral equation $(I - T)u = k$ has a unique solution $u$, where $I$ is the identity operator in $C(U)$ and $k \in B_0(U)$ such that $k|\partial U \equiv 0$, $0 \leq k \leq h$ on $U$, and $k(x) = h(x)$. Clearly $u \in B_{n-1}p(U)$, $u \equiv k$ on $(\partial U) \cup (U^* \cap \Delta)$, and $0 \leq u \leq k$ on $U$. Extend $u$ to $R$ by setting $u|R - U \equiv 0$ and then construct $u_0 \in B_{n-1}p(R)$ such that $u \leq u_0 \leq pu$ on $R$. Here $pu$ is the harmonic projection of $u$ on $R$. Note that $u_0(x) = h(x)$, $u_0 \equiv 0$ on $\Delta - U^*$, and $u_0 \leq h$ on $R$. 
Let \( \{ R_i \}^\infty_{i=1} \) be a regular exhaustion of \( R \) and take \( w_i \in B_p(R) \) such that \( w_i \equiv u_0 \) on \( R - R_i \). In view of \( \Delta u_0 = n^{-1} p u_0 \leq pu_0 \) on \( R \), it is not difficult to see that \( 0 \leq w_i \leq w_{i+1} \leq u_0 \) on \( R \) and \( \| w_i \| = \| u_0 \| \). By Harnack's principle for \( \Delta u = pu \), the sequence \( \{ w_i \} \) converges, uniformly on a compact subset of \( R \), to a function \( w \in B_p(R) \), such that \( 0 \leq w \leq u_0 \) on \( R \) and \( \| w \| = \| u_0 \| \). Since \( w \equiv 0 \) on \( \Delta - U^* \), we conclude that

\[
\max_{U^* \cap \Delta} w = \| w \| = \| u_0 \| = \max_{U^* \cap \Delta} u_0(x) = h(x) > \epsilon.
\]

In view of \( w \in F_h \), \( \max_{U^* \cap \Delta} w \geq \max_{U^* \cap \Delta} u > \epsilon \). But this contradicts our choice of \( U \) : \( v < \epsilon \) on \( U^* \).

This completes the proof of our theorem.

3. As an application of our theorem we would like to mention its contribution to the comparison problem of the spaces \( B_p(R) \) for various densities \( p(x) \). In this vein an elegant result of Royden [7] states: if \( p(x) \) and \( q(x) \) are two densities such that for some constant \( \alpha > 1 \), \( \alpha^{-1} p(x) \leq q(x) \leq \alpha p(x) \) off some compact subset of \( R \), then \( B_p(R) \) and \( B_q(R) \) are isometric. Later Nakai [4] found another important criterion: the same conclusion holds if \( \int_R |p(x) - q(x)| \, dx < \infty \).

The following result considerably sharpens their conclusions in view of the fact that \( \Delta \) is topologically "small" in \( R^* - R \).

**Corollary.** Banach spaces \( B_p(R) \) and \( B_q(R) \) are isometrically isomorphic in each of the following cases:

(i) there exists a constant \( \alpha > 1 \) such that \( \alpha^{-1} p(x) \leq q(x) \leq \alpha p(x) \) in some open neighborhood \( U^* \) of \( \Delta \) in \( R^* \);

(ii) there exists an open neighborhood \( U^* \) of \( \Delta \) in \( R^* \) such that \( \int_{U^* \cap R} |p(x) - q(x)| \, dx < \infty \).

**Proof.** In case (i), it is easy to see that \( \beta(p) = \beta(q) \). Now assume that condition (ii) holds. Contrary to our conclusion, suppose that there exists a point \( a \in \beta(p) - \beta(q) \). In this case there exists a \( u \in B_p(R) \) such that \( 0 < u < 1 \) on \( R \) and \( u(a) = 1 \). Since \( a \not\in \beta(q) \), \( u(a) = 0 \) for \( v \in B_q(R) \). Choose an open neighborhood \( V^* \) of \( a \) in \( R^* \) such that

\[
\sup_{x \in V} \int_V G_V(x, y)p(y) \, dy < \infty \quad \text{and} \quad \int_V |p(x) - q(x)| \, dx < \infty,
\]

where \( V = V^* \cap R \). It can be shown from the second inequality that

\[
\int_V G_V(x, y)p(y) - q(y) \, dy < \infty
\]
for each $x \in V$. For a "regular" exhaustion $\{\Omega_n\}_{n=1}^\infty$ of $V$ construct $v_n$ on $V$ such that $v_n \in B_q(\Omega_n)$ and $v_n \equiv u$ on $V - \Omega_n$. Then we can write

$$u(x) = h_n(x) - \int_{\Omega_n} G_n(x, y)p(y)v_n(y) \, dy,$$

$$v_n(x) = h_n(x) - \int_{\Omega_n} G_n(x, y)q(y)v_n(y) \, dy$$

on $\Omega_n$, where $h_n \in B_0(\Omega_n)$ with $h_n|\partial\Omega_n = u$ and $G_n(x, y)$ is the Green's function on $\Omega_n$. Since $0 \leq h_n \leq 1$ and $0 \leq v_n \leq 1$, we may assume that $h_n \to h \in B_0(V)$ and $v_n \to v \in B_q(V)$, uniformly on compact subsets of $V$. In view of

$$|u(x) - v_n(x)| \leq \int_{\Omega_n} G_n(x, y)|q(y) - p(y)|v_n(y) \, dy$$

$$+ \int_{\Omega_n} G_n(x, y)p(y)|v_n(y) - u(y)| \, dy$$

$$\leq \int_{\Omega_n} G_v(x, y)[|q(y) - p(y)| + p(y)] \, dy$$

we conclude that

$$|u(x) - v(x)| \leq \int_V G_v(x, y)[|q(y) - p(y)| + p(y)] \, dy$$

on $V$ and therefore on $V \cup \{a\}$. Note that all three functions in the above inequality have continuous extensions to $V^*$. Since the Green’s potential vanishes on $V^* \cap \Delta$, we deduce that $u(a) = u(a) = 1$, a contradiction to the fact that $a \notin \beta(q)$.

REFERENCES


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