VARIATION NORM CONVERGENCE OF FUNCTION SEQUENCES

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ABSTRACT. We prove that a pointwise convergent sequence of convex functions with a continuous limit converges with respect to the total variation norm. This yields a theorem on convexity-preserving operators which has as a corollary the result that a complex function $f$ is absolutely continuous on $[0, 1]$ if and only if the sequence $B_n(f)$ of Bernstein polynomials of $f$ converges to $f$ with respect to the total variation norm.

In this paper a theorem which is analogous to Dini’s theorem is proved:

**Theorem 1.** If $f_n$ is a pointwise convergent sequence of real-valued functions, each of which is convex on $[a, b]$ and the limit function $F$ is continuous on $[a, b]$, then the sequence $f_n$ converges to $F$ with respect to the total variation norm on $[a, b]$.

This is then used to prove

**Theorem 2.** Suppose $T_n$ is a sequence of linear operators from $AC[a, b]$ into $AC[a, b]$ such that for each $f \in AC[a, b]$, (1) $T_n(f)$ converges pointwise to $f$ on $[a, b]$; (2) if $f$ is convex on $[a, b]$ and $n$ is a nonnegative integer, $T_n(f)$ is convex on $[a, b]$; and (3) there is a number $M > 0$ such that for each nonnegative integer $n$, $\int_a^b |d(T_n(f))| \leq M \int_a^b |df|$. Then, for each $f \in AC[a, b]$, the function sequence $T_n(f)$ converges to $f$ with respect to the total variation norm.

**Corollary.** A complex-valued function $f$ is absolutely continuous on...
if and only if the sequence $B_n f$ of Bernstein polynomials of $f$ converges to $f$ with respect to the total variation norm.

An example is given to show that Theorem 1 does not extend to differences of convex functions.

Definitions and notation. A real-valued function $f$ on $[a, b]$ is said to be convex on $[a, b]$ (or simply, convex) provided that for each $[u, v] \subseteq [a, b]$ and each number $t, 0 < t < 1$,

$$f((1 - t)u + tv) \geq (1 - t)f(u) + tf(v).$$

$I$ denotes the identity function on the complex plane, and we employ the convention that $I^0$ is the constant function 1 so that for each nonnegative integer $n$ and each complex number $x$, $I^n(x) = x^n$. Hence if $f$ is a complex function on $[0, 1]$, the Bernstein polynomial sequence of $f$ is defined by

$$B_0 f = f(0) \quad \text{and} \quad B_n f = \sum_{p=0}^{n} \binom{n}{p} f \left( \frac{p}{n} \right) I^n(1 - t)^{n-p}$$

for $n$ a positive integer. For a complex function $f$ from a subset of the real numbers, $f(x-)$ and $f(x+)$ respectively denote the left and right hand limits of $f$ at $x$ in case the limit exists; if $S$ is a subset of the domain of $f$ and $f(S)$ is a bounded set, $|f|_S = \sup \{|f(x)|: x \in S\}$; if $f$ is of bounded variation on $[a, b]$, $\int_a^b |df|$ denotes the total variation of $f$ on $[a, b]$. The notation $]a, b[$ denotes the open interval $\{x: a < x < b\}$ and $(a, b)$ is reserved for an ordered pair.

1. Convex functions. We note without proof the following properties of convex functions:

If $f$ is a convex function on $[a, b]$, then

1. $f$ is continuous on $]a, b[$;
2. each of $f(a+)$ and $f(b-)$ exists and $f(a) \leq f(a+)$ and $f(b-) \geq f(b)$;
3. if, in addition, $f$ is nonconstant on $]a, b[$, then only one of the following statements is true:
   (a) $f$ is nondecreasing on $]a, b[$,
   (b) $f$ is nonincreasing on $]a, b[$,
   (c) there is a number $x_0$ in $]a, b[$ such that $f$ is nondecreasing on $[a, x_0]$ and $f$ is nonincreasing on $[x_0, b]$ and $f$ is nonconstant on $]a, x_0[$ and on $]x_0, b[$;
4. if in addition $f$ is continuous at $a$ and at $b$, then $f$ is absolutely continuous on $[a, b]$.
a continuous polygonal function is a difference of continuous convex polygonal functions.

Theorem 0. If \( f \) is a pointwise convergent sequence of convex functions on \([a, b]\) and \( F \) denotes the limit function, then

1. \( F \) is convex on \([a, b]\)
2. if \( F \) is continuous on \([a, b]\), then \( f \) converges uniformly on \([a, b]\).

Proof. Part (1) follows from the facts that a pointwise convergent function sequence converges uniformly on a finite set, and, hence, for each \([u, v] \subseteq [a, b]\) and \( t, 0 < t < 1 \), \( F((1 - t)u + tv) \geq (1 - t)F(u) + tF(v) \) must be true since \( f_n((1 - t)u + tv) \geq (1 - t)f_n(u) + tf_n(v) \) for each \( n \).

Proof of (2). There is an \( x_0 \in [a, b] \) such that \( F \) is monotone on each of \([a, x_0]\) and \([x_0, b]\). Hence it is enough to prove the theorem under the added assumption that \( F \) is nondecreasing. Suppose \( c > 0 \). There is an increasing sequence \( \{t_i\} \) with \( t_0 = a \) and \( t_k = b \) such that \( F(t_i) - F(t_{i-1}) < c, i = 1, \ldots, k \).

Let \( s_i = (t_{i-1} + t_i)/2, i = 1, \ldots, k \). If \( t_{i-1} \leq x \leq s_i \) then, since \( f_n \) is convex,

\[
(f_n(s_i) - f_n(x))/(s_i - x) \geq (f_n(t_i) - f_n(s_i))/(t_i - s_i).
\]

Since \((s_i - x)/(t_i - s_i) \leq (s_i - t_{i-1})/(t_i - s_i) = 1\), this implies

\[
 f_n(x) \leq f_n(s_i) + |f_n(t_i) - f_n(s_i)|,
\]

and hence

(A) \[
 \sup_{t_{i-1} \leq x \leq s_i} |f_n(x) - F(x)| \leq f_n(s_i) - F(t_{i-1}) + |f_n(t_i) - f_n(s_i)|.
\]

Similarly, if \( s_i < x \leq t_i \) then

\[
(f_n(x) - f_n(s_i))/(x - s_i) \leq (f_n(s_i) - f_n(t_{i-1}))/s_i - t_{i-1},
\]

whence

(B) \[
 \sup_{s_i \leq x \leq t_i} |f_n(x) - F(x)| \leq f_n(s_i) - F(x_i) + |f_n(s_i) - f_n(t_{i-1})|.
\]

Also, if \( t_{i-1} \leq x \leq t_i \), then \( f_n(x) \geq \min\{f_n(t_{i-1}), f_n(t_i)\} \), and

(C) \[
 \sup_{t_{i-1} \leq x \leq t_i} |F(x) - f_n(x)| \leq F(t_i) - \min\{f_n(t_{i-1}), f_n(t_i)\}.
\]

As \( n \to \infty \), the right-hand side of each of (A), (B) and (C) has a limit less than \( c \), and the theorem is proved.
Lemma 1.1. If $f$ is convex on $[a, b]$, $e > 0$, $g$ is convex on $[a, b]$ such that
\[ |f - g|_{[a, b]} \leq e \quad \text{and} \quad P = \frac{f(b) - f(a)}{b - a} (1 - a) + f(a), \]
then
\begin{align*}
(1) \quad \int_a^b |d(f - P)| = 2|f - P|_{[a, b]} \\
(2) \quad \int_a^b |d(g - P)| \leq \int_a^b |d(f - P)| + 4e.
\end{align*}

Proof. (1) follows immediately from the unproved assertion (3) about convex functions. To prove (2) we note that $g - P$ is convex and apply the same assertion (3) in the three separate cases. Let us consider only the case where there exists a number $x_0$ in $]a, b[$ such that $g - P$ is nondecreasing on $[a, x_0]$, nonincreasing on $[x_0, b]$ and nonconstant on each of $]a, x_0[$ and $]x_0, b[$. Thus
\begin{align*}
\int_a^b |d(g - P)| &= 2(g - P)(x_0) - (g - P)(a) - (g - P)(b) \\
&= 2g(x_0) - 2P(x_0) - g(a) + f(a) - g(b) + f(b) \\
&\leq 2f(x_0) + e - 2P(x_0) + e + e \\
&= 2f(x_0) - P(x_0) + 4e \leq 2|f - P|_{[a, b]} + 4e \\
&= \int_a^b |d(f - P)| + 4e.
\end{align*}

We omit proof of the other two cases.

Lemma 1.2, which follows, was proved independently by the author for convex functions only. It follows immediately from a result of J. R. Edwards and S. G. Wayment [1, p. 254] on absolutely continuous functions and the fact that a continuous convex function is absolutely continuous.

Lemma 1.2. If $F$ is a continuous convex function on $[a, b]$ and $c > 0$, then there is an increasing sequence $\{t_n\}$ with $t_0 = a$ and $t_n = b$ such that if $P$ is the function on $[a, b]$ defined by
\[ P(x) = \frac{F(t_{p+1}) - F(t_p)}{t_{p+1} - t_p} (x - t_p) + F(t_p) \quad \text{for} \ x \in [t_p, t_{p+1}], \]
then $\int_a^b |d(F - P)| < c$.

Theorem 1. If $f_\ast$ is a pointwise convergent sequence of functions on
Proof. Suppose the hypothesis and let $c > 0$. There is an increasing sequence \{t_p\}_0^n$ with $t_0 = a$ and $t_n = b$ such that if $P$ is the function as defined in Lemma 1.2 for $F$, then $\int_a^b |d(F - P)| < c/4$. Let $e = c/(8n)$. There is a positive integer $N$ such that if $q$ is an integer, $q > N$, then $|f_q - F|_{[a, b]} < e$ by Theorem 0. For each integer $q > N$

$$\int_a^b |d(f_q - F)| \leq \int_a^b |d(f_q - P)| + \int_a^b |d(P - F)|$$

$$< \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} |d(f_q - P)| + \frac{c}{4}.$$  

But Lemma 1.1 and the fact that $|f_q - F|_{[t_p, t_{p+1}]} \leq |f_q - F|_{[a, b]} < e$ imply that for each integer $p$, $0 \leq p \leq n - 1$,

$$\int_{t_p}^{t_{p+1}} |d(f_q - P)| \leq \int_{t_p}^{t_{p+1}} |d(F - f_q)| + 4e.$$  

Whence we see that

$$\sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} |d(f_q - P)| \leq \sum_{p=0}^{n-1} \left\{ \int_{t_p}^{t_{p+1}} |d(F - P)| + 4e \right\}$$

$$= \int_a^b |d(F - P)| + 4ne = \int_a^b |d(f - P)| + \frac{c}{2} < \frac{3c}{4}.$$  

Thus $\int_a^b |d(f_q - F)| < c$ for each integer $q > N$.

Corollary. If $F$ is a continuous convex function on $[0, 1]$, then the sequence $B_nF$ of Bernstein polynomials of $F$ converges to $F$ with respect to the total variation norm on $[0, 1]$.

Proof. This is an immediate consequence of the well-known facts that since $F$ is continuous, $B_nF$ converges uniformly to $F$ and that for each non-negative integer, $n$, $B_nF$ is convex on $[0, 1]$; cf. Lorentz [3, p. 5 and p. 23 resp.].

Remark. Theorem 1 does not extend to sequences of differences of convex functions, as may be seen from the following example: let $f_0$ be a sequence of functions on $[0, 1]$ such that for each positive integer $n$, and nonnegative integer $p < 2^n$
Each function \( f_n \) is a continuous polygonal function with \( \int_a^b |f_n| = 1 \), and the sequence \( f \) converges uniformly to the constant function 0.

2. Absolutely continuous functions. A function \( f \) is said to be absolutely continuous on \([a, b]\) provided that for each \( c > 0 \) there is a positive number \( d \) such that if \( \{[u_p, v_p]\}_0^n \) is a sequence of nonoverlapping subintervals of \([a, b]\) with \( \sum_{p=0}^{n} (v_p - u_p) < d \), then \( \sum_{p=0}^{n} |f(v_p) - f(u_p)| < c \). It is well known [1] that the class \( AC[a, b] \) of all absolutely continuous real-valued functions on \([a, b]\) is complete with respect to the total variation norm and that the polygonal functions form a dense subset thereof.

Theorem 2. Suppose \( T_\ast \) is a sequence of linear operators from \( AC[a, b] \) into \( AC[a, b] \) such that for each \( f \) in \( AC[a, b] \), (1) \( T_\ast f \) converges pointwise to \( f \) on \([a, b]\); (2) if \( f \) is convex on \([a, b]\) and \( n \) is a nonnegative integer, \( T_n f \) is convex on \([a, b]\); and (3) there is a number \( M > 0 \) such that for each nonnegative integer \( n \), \( \int_a^b |d(T_n f)| \leq M \int_a^b |d f| \). Then, for each \( f \in AC[a, b] \), the function sequence \( T_n f \) converges to \( f \) with respect to the total variation norm.

Proof. Let \( \mathcal{B} \) denote the set of all real-valued functions \( f \) on \([a, b]\) such that \( T_n f \) converges to \( f \) with respect to the total variation norm. \( \mathcal{B} \) is closed with respect to the total variation norm, for if \( F \) is the limit with respect to the total variation norm of a sequence \( f_n \) with values in \( \mathcal{B} \), then

\[
\int_a^b |d(F - T_n F)| \leq \int_a^b |d(F - f_k)| + \int_a^b |d(f_k - T_n(f_k))| + \int_a^b |d(T_n(f_k) - T_n(F))|.
\]

But from part (3) of the hypothesis we have that

\[
\int_a^b |d(T_n(f_k) - T_n(F))| = \int_a^b \left| d(T_n(f_k - F)) \right| \leq M \int_a^b |d(f_k - F)|.
\]

Thus

\[
\int_a^b |d(F - T_n(F))| \leq (M + 1) \int_a^b |d(f_k)| + \int_a^b |d(f_k - T_n(f_k))|,
\]

from which it is clear that \( \mathcal{B} \) is closed with respect to the total variation norm.

If \( P \) is a polygonal function on \([a, b]\) then \( P \) is a difference of continuous convex functions, say \( P = h - k \). But for each nonnegative integer \( n \),
\( T_n P = T_n (h - k) = T_n (h) - T_n (k); \) whence by Theorem 1 and parts (1) and (2) of the hypothesis, \( P \) must belong to \( \mathcal{B} \). Thus \( \mathcal{B} = AC[a, b] \) since \( \mathcal{B} \) contains all polygonal functions and is closed with respect to the total variation norm.

**Corollary.** A complex-valued function \( f \) is absolutely continuous on \( [0, 1] \) if and only if the sequence \( B_n f \) of Bernstein polynomials of \( f \) converges to \( f \) with respect to the total variation norm.

**Proof.** Let us note that if \( f \) is a complex-valued absolutely continuous function on \( [0, 1] \), then each of \( \text{Re} f \) and \( \text{Im} f \) is absolutely continuous; and if \( n \) is a nonnegative integer, \( B_n f = B_n \text{Re} f + i B_n \text{Im} f \). Thus it is sufficient to suppose \( f \) to be real valued, and we do so. Since any polynomial is absolutely continuous on \( [0, 1] \), then any function \( f \) on \( [0, 1] \) such that \( B_n f \) converges to \( f \) with respect to the total variation norm must perforce be absolutely continuous. Theorem 2 yields the converse.

**Comment.** The corollary to Theorem 2 has been obtained independently by G. G. Johnson, who used methods different from ours. While the results herein give no estimate on the size of \( \int_0^1 |d(F - B_n(F))| \), they offer an extension of a result of W. Hoeffding [2, p. 349] that: If \( f \) is a continuous convex function such that \( \int_0^1 \sqrt{2}(1 - t)^{\frac{1}{2}} \, d|f'| \) exists, then \( B_n f \) converges with respect to the total variation norm. Some applications of these results to moment problems will appear in a subsequent paper.

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**REFERENCES**


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