THE MINIMUM MODULUS OF POLYNOMIALS

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ABSTRACT. In answer to a problem of Erdős and Littlewood we produce an nth degree polynomial, \( P(z) \), with coefficients bounded by 1 satisfying \( |P(z)| > C \sqrt{n} \) for all \( z \) on \( |z| = 1 \) (\( C \) is a positive absolute constant).

Littlewood and Erdős, independently, asked whether polynomials, \( P(z) \), of degree \( n \) could exist having all coefficients bounded by 1 and satisfying \( \min_{|z| = 1} |P(z)| \geq C \sqrt{n} \) (\( C \) a fixed positive constant).

Clunie [2] gave a very ingenious construction of a polynomial purporting to do this job, but it was based on a result of Littlewood’s [3] which later proved erroneous. Littlewood claimed that, as \( r \to 1^- \),

\[
\min_{|z| = r} \left| \sum_{n=1}^{\infty} n^{in} z^n \right| = \Omega(1 - r)^{-1/2},
\]

but his reasoning had a flaw which was discovered by Erdős and Carroll. Indeed a careful examination of his method shows the very opposite, that

\[
\min_{|z| = r} \left| \sum_{n=1}^{\infty} n^{in} z^n \right| = o(1 - r)^{-1/2}.
\]

In this note we give an extremely simple construction of a polynomial which does have the desired properties.

Consider the function \( f(\theta) \) defined as \( \exp(in \theta^2) \) in \( [-\pi, \pi] \) and extended to have period \( 2\pi \). (\( \delta \) is a small but fixed positive number.) Write \( K = [n/2] \), let \( t(\theta) \) be the \( K \)th Cesàro partial sum of the Fourier series of \( f(\theta) \), and finally set \( P(e^{i\theta}) = \sqrt{n} \delta e^{iK\theta} \).

Clearly \( P(z) \) is a polynomial of degree \( \leq n \) and we will prove

1. The coefficients of \( P(z) \) are all bounded by 1.
2. \( |P(z)| \geq \sqrt{n}\delta(1 - 40 \delta \log \delta^{-1}) \) for all \( z \) on \( |z| = 1 \).

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To prove (1) we start with the exact formula for the coefficient of $z^j$, namely
\[
(1 - \frac{|j - K|}{K + 1}) \frac{\sqrt{n\delta}}{2\pi} \int_{-\pi}^{\pi} \exp \left(i(n\delta \theta^2 + (K - j)\theta)\right) d\theta.
\]

Since $n\delta \theta^2 + (K - j)\theta$ has second derivative equal to $2n\delta$ we may apply Lemma 4.4, p. 61 of [4] and obtain the bound
\[
\left| \int_{-\pi}^{\pi} \exp \left(i(n\delta \theta^2 + (K - j)\theta)\right) d\theta \right| \leq \frac{8}{\sqrt{2n\delta}}.
\]

Thus our coefficient is bounded by
\[
(1 - \frac{|j - K|}{K + 1}) \frac{\sqrt{n\delta}}{2\pi} \cdot \frac{8}{\sqrt{2n\delta}} < 1
\]
as required.

To prove (2) we turn to Fejér's formula
\[
t(\theta) - f(\theta) = \frac{1}{2n(K + 1)} \int_{-\pi}^{\pi} (f(\theta + u) - f(\theta)) \frac{\sin^2((K + 1)/2)}{\sin^2 (u/2)} \, du.
\]

Now $f(\theta)$ has derivative bounded by $2\pi n\delta$ so that $|f(\theta + u) - f(\theta)| \leq 2\pi n\delta |u|$. Also, of course, $|f(\theta + u) - f(u)| \leq 2$.

Using this, together with the elementary inequalities
\[
\frac{\sin^2((K + 1)/2)}{\sin^2 (u/2)} \leq (K + 1)^2 \quad \text{and} \quad \frac{\sin^2((K + 1)/2)}{\sin^2 (u/2)} \leq \frac{1}{\sin^2 (u/2)} \leq \frac{n^2}{u^2},
\]
we obtain
\[
(K + 1)|t(\theta) - f(\theta)|
\leq \int_{0}^{\pi/(K + 1)} 2n\delta u \cdot (K + 1)^2 \, du + \int_{\pi/(K + 1)}^{1/\pi n\delta} 2n\delta u \cdot \frac{n^2}{u^2} \, du + \int_{1/\pi n\delta}^{\pi} \frac{2n\delta}{u^2} \, du
= 2n^2\delta \log \frac{(K + 1)e^{3/2}}{\pi^2 n\delta} - 2 \leq 2n^2\delta \log \frac{e^{3/2}}{\pi^2 \delta} - 1 < 20n\delta \log \frac{1}{\delta}.
\]

Since $K + 1 > n/2$ we may thereby conclude that
\[
|t(\theta)| \geq 1 - |t(\theta) - f(\theta)| > 1 - 40\delta \log \delta^{-1}
\]
and (2) follows immediately.

There is an extra dividend issuing from this construction. If we simply set $\delta = 1/400$, for example, then we do obtain a polynomial which does the Littlewood-Clunie job. Notice, however, that $t(\theta)$ is automatically bounded by 1, the bound for $f(\theta)$, so that we obtain the upper bound $|P(z)| \leq \sqrt{n\delta}$. 

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Hence if we make $\delta$ very much smaller, our polynomial will have the additional property that

$$\text{Max} |P(z)| \leq (1 + \epsilon) \min |P(z)|.$$  

The constant we obtain is quite a poor one, but it can be improved using the method given in [1]. Indeed we can produce a $P(z)$ such that $|P(z)| \geq .395 \sqrt{n}$ for large $n$.

REFERENCES


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