GLOBAL DIMENSION OF DIFFERENTIAL OPERATOR RINGS

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ABSTRACT. This paper is concerned with finding the global homological dimension of the ring of differential operators $R[\theta]$ over a differential ring $R$ with a single derivation. Examples are constructed to show that $R[\theta]$ may have finite dimension even when $R$ has infinite dimension. For a commutative noetherian differential algebra $R$ over the rationals, with finite global dimension $n$, it is shown that the global dimension of $R[\theta]$ is the supremum of $n$ and one plus the projective dimensions of the modules $R/P$, where $P$ ranges over all prime differential ideals of $R$. One application derives the global dimension of the Weyl algebra over a commutative noetherian ring $S$ of finite global dimension, where $S$ either is an algebra over the rationals or else has positive characteristic.

1. Introduction. Throughout this paper, the term differential ring will refer to an associative ring $R$ with unit together with a specified derivation $\delta$ on $R$. The ring of differential operators over $R$ (also called the Ore-extension of $R$ with respect to $\delta$) is additively the group of polynomials over $R$ in an indeterminate $\theta$, with multiplication subject to the requirement $\delta a = a\theta + \delta a$ for all $a \in R$. We denote this ring by $R[\theta]$, or by $(R, \delta)[\theta]$ if $\delta$ needs to be emphasized.

Most arguments involving right-hand properties of $R[\theta]$ can be used, with appropriate changes of sign, for the corresponding left-hand properties. The precise relationship is given by the following proposition, whose proof is routine.

Proposition 1. Let $R$ be a differential ring with derivation $\delta$. If $R^0$ denotes the opposite ring of $R$ together with the derivation $-\delta$, then $R^0[\theta]$ is isomorphic to the opposite ring of $R[\theta]$.

In particular, Proposition 1 can be used to show that all of the results in
this paper on the right global dimension of \( R[\theta] \) carry over to the left global dimension, hence we shall not state the left-hand versions explicitly.

2. General differential rings. In considering the global dimension of \( R[\theta] \) several authors [1], [3], [4], [10] have stated as "well-known" the inequalities

\[
\text{r. gl. dim. } R \leq \text{r. gl. dim. } R[\theta] \leq 1 + \text{r. gl. dim. } R.
\]

While the right-hand inequality is true in general, the left-hand one may fail if \( \text{r. gl. dim. } R \) is allowed to be infinite, and we present examples of this at the end of the section. For clarity, therefore, we now derive the valid cases of these inequalities.

Given a module \( A \) over a ring \( S \), we use \( \text{pd}_S(A) \) and \( \text{wd}_S(A) \) to denote the projective and weak dimensions of \( A \).

**Proposition 2.** Let \( R \) be a differential ring. If \( A \) is any right \( R[\theta] \)-module, then

\[
\text{pd}_R(A) < \text{pd}_R[\theta](A) < 1 + \text{pd}_R(A),
\]

and

\[
\text{wd}_R(A) < \text{wd}_R[\theta](A) < 1 + \text{wd}_R(A).
\]

**Proof.** Inasmuch as \( (R[\theta])_R \) is free, any projective resolution for \( A_R[\theta] \) is also a projective resolution for \( A_R \), whence \( \text{pd}_R(A) \leq \text{pd}_R[\theta](A) \). As in the case of polynomials [6, Part III, Theorem 6], there exists an exact sequence \( 0 \rightarrow A \otimes_R R[\theta] \rightarrow A \otimes_R R[\theta] \rightarrow A \rightarrow 0 \) of right \( R[\theta] \)-modules, from which we infer that \( \text{pd}_R[\theta](A) \leq 1 + \text{pd}_R(A) \).

The inequalities for weak dimension are proved in the same way.

**Proposition 3.** If \( R \) is any nonzero differential ring, then

\[
1 \leq \text{r. gl. dim. } R[\theta] \leq 1 + \text{r. gl. dim. } R.
\]

In case \( \text{r. gl. dim. } R < \infty \), then also

\[
\text{r. gl. dim. } R \leq \text{r. gl. dim. } R[\theta].
\]

**Proof.** The inequality \( \text{r. gl. dim. } R[\theta] \leq 1 + \text{r. gl. dim. } R \) is clear from Proposition 2. Observing that \( R[\theta]/\theta R[\theta] \) cannot be isomorphic to a right ideal of \( R[\theta] \), we see that \( \text{r. gl. dim. } R[\theta] \geq 1 \).

Now assume that \( \text{r. gl. dim. } R = n < \infty \), and choose a right \( R \)-module \( A \) with \( \text{pd}_R(A) = n \). There is an obvious exact sequence \( 0 \rightarrow A \rightarrow A \otimes_R R[\theta] \rightarrow B \rightarrow 0 \), and certainly \( \text{pd}_R(B) \leq n \), hence we infer from the long exact sequence.
for Ext that $\text{pd}_R(A \otimes_R R[\theta]) = n$. In view of Proposition 2, it now follows that the right global dimension $R[\theta] \geq n$.

We now give an example of a differential ring $R$ with right global dimension $R = \infty$ and right global dimension $R[\theta] = 1$.

Choose a field $F$ of characteristic 2 and consider the derivation $\frac{\partial}{\partial x}$ on $F[x]$. The ideal $(x^2)$ is a differential ideal because of characteristic 2, hence we obtain a differential ring $R = F[x]/(x^2)$. As an $F$-algebra, $R$ has a basis $1, w$ such that $w^2 = 0$ and $\delta w = 1$. Observing an exact sequence $0 \to wR \to R \to wR \to 0$, we infer that $\text{gl. dim. } R = \infty$.

Given any right $R[\theta]$-module $A$, we may choose a basis $\{b_\alpha\}$ over $F$ for the vector space $B = \{b \in A \mid bw = 0\}$. Set $a_\alpha = b_\alpha \theta$ for all $\alpha$, and note that $a_\alpha w = b_\alpha$. It is routine to check that $A_R$ is free with basis $\{a_\alpha\}$, hence we obtain $\text{pd}_R R[\theta](A) \leq 1$ from Proposition 2, and thus the right global dimension $R[\theta] = 1$.

This example may be extended somewhat as follows. Set $R_1 = R$ as above, and for $n > 1$ set $R_n = R[x_1, \ldots, x_{n-1}]$ and extend $\delta$ to a derivation of $R_n$ by defining $\delta x_i = 0$ for all $i$. This choice of $\delta$ ensures that $x_i$ all commute with $\theta$ in $R_n[\theta]$, whence $R_n[\theta] \cong R[\theta][x_1, \ldots, x_{n-1}]$ and so the right global dimension $R_n[\theta] = n$. We summarize these examples as follows:

For each positive integer $n$, there exists a commutative noetherian differential ring $R_n$ such that $\text{gl. dim. } R_n = \infty$ and the right global dimension $R_n[\theta] = n$.

3. Commutative noetherian differential rings. For some of the results in this section, we require that $R$ be an algebra over the rationals, in which case we follow Kaplansky [5] in using the term Ritt algebra to denote a commutative differential ring which is an algebra over the rationals.

**Lemma 4.** Let $R$ be a noetherian Ritt algebra. If $A$ is any nonzero finitely generated right $R[\theta]$-module, then there exists a chain of $R[\theta]$-submodules $A_0 = 0 < A_1 < \cdots < A_k = A$ and a collection of prime ideals $P_1, \ldots, P_k$ of $R$ such that for each $i$, either $A_i/A_{i-1} \cong R[\theta]/P_i R[\theta]$ or else $P_i$ is a differential ideal and $A_i/A_{i-1}$ is a torsion-free $(R/P_i)$-module.

**Proof.** As observed in [1, p. 68], $R[\theta]$ is right noetherian, and so $A$ has ACC on $R[\theta]$-submodules. Thus it suffices to show that there exists a nonzero $R[\theta]$-submodule $B$ of $A$ and a prime ideal $P$ in $R$ satisfying one of the two relationships described.

Choose a nonzero $x \in A$ whose $R$-annihilator $P = \{r \in R \mid xr = 0\}$ is maximal among the $R$-annihilators of nonzero elements of $A$, and set $B = xR[\theta]$. According to [7, Theorem 6], $P$ is a prime ideal of $R$. If $P$ is a differential ideal, then $R[\theta]P = PR[\theta]$ and we obtain $BP = 0$. In this case the maximality
of $P$ now ensures that $B_{R/P}$ is torsion-free. Clearly $B \cong R[\theta]/J$ for some right ideal $J$ of $R[\theta]$ such that $PR[\theta] \subseteq J$ and $J \cap R = P$. If $P$ is not a differential ideal, then an easy argument due to Hart in [4, Lemma 2.4] shows that $J = PR[\theta]$.

**Theorem 5.** Let $R$ be a noetherian Ritt algebra with gl. dim. $R = n < \infty$. If $\text{pd}_R(R/P) < n$ for all prime differential ideals $P$ of $R$, then r. gl. dim. $R[\theta] = n$.

**Proof.** If $n = 0$, then $R$ must contain a prime ideal $P$ which is generated by an idempotent $e$. Differentiating the identities $e^2 = e$ and $(1 - e)^2 = 1 - e$, we find that $\delta e = 0$ and so $P$ is a differential ideal. But $\text{pd}_R(R/P) < 0$ is impossible, hence we must have $n \neq 0$.

We have r. gl. dim. $R[\theta] \geq n$ by Proposition 3, hence it suffices to prove that $\text{pd}_R[\theta](A) \leq n$ for any nonzero finitely generated $A_R[\theta]$. Since $R[\theta]$ is right noetherian, it is enough to show that $\text{wd}_R[\theta](A) \leq n$. In view of Lemma 4, we may also assume that there is a prime ideal $P$ in $R$ such that either $A \cong R[\theta]/PR[\theta]$ or else $P$ is a differential ideal and $A$ is a torsion-free $(R/P)$-module.

If $A \cong R[\theta]/PR[\theta]$, then obviously $\text{wd}_R[\theta](A) \leq \text{wd}_R(R/P) \leq n$.

Now assume that $P$ is a differential ideal and that $A$ is a torsion-free $(R/P)$-module. Then $A_R$ can be embedded in some direct product $T$ of copies of the quotient field of $R/P$, and since $R/P$ is noetherian we see that $T_{R/P}$ is flat. We are given $\text{pd}_R(R/P) \leq n - 1$, and we infer from [2, Proposition 4.1.2, p. 117] that $\text{wd}_R(T) \leq n - 1$ also. There is an exact sequence $0 \to A \to T \to C \to 0$, and certainly $\text{wd}_R(C) \leq n$, hence it follows from the long exact sequence for Tor that $\text{wd}_R(A) \leq n - 1$. By Proposition 2, $\text{wd}_R[\theta](A) \leq n$.

**Lemma 6.** Let $R$ be a commutative noetherian differential ring with gl. dim. $R = n < \infty$. If $R$ is local with maximal ideal $M$, and if $M$ is a differential ideal, then r. gl. dim. $R[\theta] = n + 1$.

**Proof.** We already have r. gl. dim. $R[\theta] \leq n + 1$ by Proposition 3. According to [6, Part III, Theorems 12, 13], $R$ has classical Krull dimension $n$, and by [8, (h)] $R$ must also have Krull dimension $n$ in the sense of [8].

Inasmuch as $M$ is a differential ideal, $R/M$ is a differential ring and $MR[\theta]$ is a two-sided ideal of $R[\theta]$ with $R[\theta]/MR[\theta] \cong (R/M)[\theta]$. Observing that the right Krull dimension of $(R/M)[\theta]$ is at least 1, we see from [1, Lemma 2.c.1] that the right Krull dimension of $R[\theta]$ is at least $n + 1$. 

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Now $R[\theta]$ is a filtered noetherian ring whose associated graded ring is the commutative regular noetherian ring $R[x]$. According to a theorem of Roos in [11] (quoted in [1, p. 78]), the right Krull dimension of $R[\theta]$ is at most r. gl. dim. $R[\theta]$, hence we obtain r. gl. dim. $R[\theta] \geq n + 1$.

**Lemma 7.** Let $R$ be a commutative differential ring, $S$ any multiplicative subset of $R$.

(a) $\delta$ induces a derivation on $R_S$ by the rule $\delta(r/s) = [(\delta r)s - r(\delta s)]/s^2$.

(b) The natural map $\phi: R[\theta] \rightarrow R_S[\theta]$ makes $R_S[\theta]$ into a flat left $R[\theta]$-module such that the multiplication map $R_S[\theta] \otimes_R[\theta] R_S[\theta] \rightarrow R_S[\theta]$ is an isomorphism.

(c) r. gl. dim. $R_S[\theta] \leq$ r. gl. dim. $R[\theta]$.

**Proof.** (a) is an easy verification.

(b) We check that $\ker \phi = \{x \in R[\theta] \mid xs = 0 \text{ for some } s \in S\}$, that each element of $\phi(S)$ is invertible in $R_S[\theta]$, and that every element of $R_S[\theta]$ may be written in the form $(\phi x)(\phi s)^{-1}$ for suitable $x \in R[\theta]$, $s \in S$. Thus $R_S[\theta]$ is just the classical localization of $R[\theta]$ with respect to the multiplicative set $S$, and the required properties follow as in the commutative case.

(c) According to (b), $R_S[\theta]$ is a right localization of $R[\theta]$ in the sense of [12], hence this inequality follows from [12, Corollary 1.3].

**Theorem 8.** Let $R$ be a commutative noetherian differential ring with gl. dim. $R = n < \infty$. If $R$ contains a prime differential ideal $P$ such that $\text{pd}_R(R/P) = n$, then r. gl. dim. $R[\theta] = n + 1$.

**Proof.** We already have r. gl. dim. $R[\theta] \leq n + 1$ by Proposition 3. According to [6, Part III, Theorem 11], gl. dim. $R_P \leq n$. There is a natural exact sequence $0 \rightarrow R/P \rightarrow R_P/PR_P \rightarrow A \rightarrow 0$, and we have $\text{wd}_R(R/P) = n$, $\text{wd}_R(A) \leq n$, hence it follows from the long exact sequence for Tor that $\text{wd}_R(R_P/PR_P) = n$. Because $(R_P)_R$ is flat, the weak dimension of $R_P/PR_P$ over $R_P$ must be at least $n$, and we conclude that gl. dim. $R_P = n$.

Now $R_P$ is a differential ring as in Lemma 7, and $PR_P$ is a differential ideal of $R_P$ because $P$ is a differential ideal of $R$, hence we obtain r. gl. dim. $R_P[\theta] = n + 1$ from Lemma 6. Therefore r. gl. dim. $R[\theta] \geq n + 1$, by Lemma 7.

**Corollary 9.** Let $R$ be a commutative noetherian differential ring with gl. dim. $R = n < \infty$, and assume that $R$ has prime characteristic $p$. If $R$ contains a prime ideal $Q$ such that $\delta^p(Q) \subseteq Q$ and $\text{pd}_R(R/Q) = n$, then r. gl. dim. $R[\theta] = n + 1$.
Proof. Because of characteristic \( p \), it follows from Leibnitz' rule that \( \delta^p \) is a derivation on \( R \). Likewise, we obtain \( \theta^p a = a \theta^p + \delta^p a \) for all \( a \in R \), whence the set \( R[\theta^p] \) is a subring of \( R[\theta] \) isomorphic to \( (R, \delta^p)[\theta] \). Since \( Q \) is a prime differential ideal in \( (R, \delta^p) \), Theorem 8 says that r. gl. dim. \( R[\theta^p] = n + 1 \). Observing that \( R[\theta] \) is a free right and left \( R[\theta^p] \)-module, we obtain r. gl. dim. \( R[\theta] \geq n + 1 \) from [1, Lemma 2.b.4], and by Proposition 3 we are done.

Combining Theorems 5 and 8 immediately yields the following formula:

**Theorem 10.** Let \( R \) be a noetherian Ritt algebra with gl. dim \( R = n < \infty \). If \( k = \sup \{ \text{pd}_R(R/P) \mid P \) is a prime differential ideal of \( R \} \), then r. gl. dim. \( R[\theta] = \max \{ n, k + 1 \} \).

4. Applications and remarks. (a) The Weyl algebra \( A_1(S) \) over a ring \( S \) is just \( S[x][\theta] \), where we use the derivation \( \delta = \partial/\partial x \) on \( S[x] \). If \( S \) is a field of characteristic \( 0 \), then r. gl. dim. \( A_1(S) = 1 \), while if \( S \) is a field of positive characteristic, then r. gl. dim. \( A_1(S) = 2 \) [9, Theorem, p. 345]. We generalize these results as follows:

**Theorem 11.** Let \( S \) be a commutative noetherian ring with gl. dim. \( S = n < \infty \). If \( S \) is an algebra over the rationals, then r. gl. dim. \( A_1(S) = n + 1 \), while if \( S \) has positive characteristic, then r. gl. dim. \( A_1(S) = n + 2 \).

Proof. Let \( R \) denote the ring \( S[x] \) with the derivation \( \partial/\partial x \).

Assuming that \( S \) is an algebra over the rationals, let \( P \) be any differential ideal of \( R \), and set \( J = P \cap S \). It is easy to see that \( P = J R[x] \), whence \( \text{pd}_R(R/P) \leq \text{pd}_S(S/J) \leq n \). According to Theorem 5, r. gl. dim. \( A_1(S) = n + 1 \).

Now assume that \( S \) has positive characteristic. In view of [6, Part III, Theorems 11, 13], \( S \) is a regular noetherian ring, hence [7, Theorem 168] says that \( S \) is a finite direct product of domains. Thus we need only consider the case when \( S \) is a domain, and here the characteristic of \( S \) must be a prime \( p \).

In view of [7, Exercise 7, p. 53], we must have \( \text{pd}_S(S/P) = n \) for some prime ideal \( P \) of \( S \). Let \( Q = P + xR \), which is a prime ideal of \( R \) such that \( R/Q \cong S/P \). Now \( \text{pd}_R(R/Q) = n + 1 \) by [6, Part III, Theorem 3], and it is clear that \( \delta^p(Q) \subseteq Q \), hence Corollary 9 says that r. gl. dim. \( A_1(S) = n + 2 \).

(b) For any ring \( S \), one can define a ring \( F_1(S) = S[[x]][\theta] \) using the derivation \( \partial/\partial x \) on \( S[[x]] \). Proceeding as in Theorem 11, we obtain the following results, of which the first part is a special case of [1, Theorem 4.2].

**Theorem 12.** Let \( S \) be a commutative noetherian ring with gl. dim. \( S = \)}
If $S$ is an algebra over the rationals, then $\text{r. gl. dim. } F_1(S) = n + 1$, while if $S$ has positive characteristic, then $\text{r. gl. dim. } F_1(S) = n + 2$.

(c) [4, Theorem 2.6] says that if $R$ is a commutative noetherian integral domain (not a field) with finite Krull dimension, and if $\delta$ is a derivation on $R$ such that $R[\delta]$ is a simple ring, then $\text{r. gl. dim. } R[\delta] = \text{gl. dim. } R$. As noted in [4, Lemma 2.4], it follows from these hypotheses that $R$ is a Ritt algebra and that the only differential ideals in $R$ are $0$ and $R$. Thus for the case $\text{gl. dim. } R < \infty$, this theorem is a consequence of Theorem 5. For the case $\text{gl. dim. } R = \infty$, however, the proof of this theorem relies on the inequality $\text{gl. dim. } R \leq \text{r. gl. dim. } R[\delta]$ and so is open to question.

(d) [3, Theorem 2] is another theorem which relies on the inequality $\text{gl. dim. } R \leq \text{r. gl. dim. } R[\delta]$ in the infinite-dimensional case and so is open to question. In the finite-dimensional case, this result is an easy consequence of Theorem 8.

(e) We conclude by noting that Theorems 5 and 10 may fail in cases of positive characteristic. For example, let $F$ be any algebraically closed field of positive characteristic, and set $R = F[x], \delta = \partial/\partial x$. Since the maximal ideals of $R$ are generated by linear polynomials, none of them can be differential ideals. Thus we have $\text{pd}_R(R/P) = 0 < \text{gl. dim. } R$ for all prime differential ideals $P$ of $R$, whereas Theorem 11 says that
\[\text{r. gl. dim. } R[\delta] = 2 > \text{gl. dim. } R.\]

Perhaps Theorems 5 and 10 may be made to hold in general by using more differential ideals $P$ than just the prime ones.

Added in proof.

(f) Some results related to those in this paper have appeared in S. M. Bhatwadekar, *On the global dimension of Ore-extensions*, Nagoya Math. J. 50 (1973), 217–225. Bhatwadekar's Theorem 2.3 gives the rational case of our Theorem 11, and his Theorem 1.1 can be obtained as a direct consequence of our Theorem 5.


(h) In a sequel to the present paper, we generalize the results given here to the case of a commutative noetherian ring with a finite collection of commuting derivations. Also, we answer the question raised in Remark (e): Theorem 10 holds for commutative noetherian differential rings $R$ with
gl. dim. $R < \infty$, provided $k$ is set equal to the supremum of $\text{pd}_R(R/P)$ where $P$ ranges over all primary differential ideals of $R$.

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