PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September, 1974

GLOBAL DIMENSION OF DIFFERENTIAL OPERATOR RINGS

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ABSTRACT. This paper is concerned with finding the global homological dimension of the ring of differential operators $R[\theta]$ over a differential ring R with a single derivation. Examples are constructed to show that $R[\theta]$ may have finite dimension even when R has infinite dimension. For a commutative noetherian differential algebra R over the rationals, with finite global dimension n, it is shown that the global dimension of $R[\theta]$ is the supremum of n and one plus the projective dimensions of the modules R/P, where P ranges over all prime differential ideals of R. One application derives the global dimension of the Weyl algebra over a commutative noetherian ring S of finite global dimension, where S either is an algebra over the rationals or else has positive characteristic.

1. **Introduction.** Throughout this paper, the term *differential ring* will refer to an associative ring R with unit together with a specified derivation δ on R. The *ring of differential operators* over R (also called the *Ore-extension* of R with respect to δ) is additively the group of polynomials over R in an indeterminate θ, with multiplication subject to the requirement $\theta a = a\theta + \delta a$ for all $a \in R$. We denote this ring by $R[\theta]$, or by $(R, \delta)[\theta]$ if δ needs to be emphasized.

Most arguments involving right-hand properties of $R[\theta]$ can be used, with appropriate changes of sign, for the corresponding left-hand properties. The precise relationship is given by the following proposition, whose proof is routine.

Proposition 1. *Let R be a differential ring with derivation δ. If R^0 denotes the opposite ring of R together with the derivation $-\delta$, then $R^0[\theta]$ is isomorphic to the opposite ring of $R[\theta]$.*

In particular, Proposition 1 can be used to show that all of the results in

Presented to the Society, January 18, 1974 under the title *Homological dimension of differential operator rings*; received by the editors April 24, 1973.

AMS (MOS) subject classifications (1970). Primary 16A60; Secondary 12H05.

Key words and phrases. Global dimension, rings of linear differential operators, differential algebra, Weyl algebras.

this paper on the right global dimension of $R[\theta]$ carry over to the left global dimension, hence we shall not state the left-hand versions explicitly.

2. General differential rings. In considering the global dimension of $R[\theta]$ several authors [1], [3], [4], [10] have stated as "well-known" the inequalities

$$\text{r. gl. dim. } R \leq \text{r. gl. dim. } R[\theta] \leq 1 + \text{r. gl. dim. } R.$$

While the right-hand inequality is true in general, the left-hand one may fai. if r. gl. dim. R is allowed to be infinite, and we present examples of this at the end of the section. For clarity, therefore, we now derive the valid cases of these inequalities.

Given a module A over a ring S, we use $\text{pd}_S(A)$ and $\text{wd}_S(A)$ to denote the projective and weak dimensions of A.

Proposition 2. *Let R be a differential ring. If A is any right $R[\theta]$-module, then*

$$\text{pd}_R(A) \leq \text{pd}_{R[\theta]}(A) \leq 1 + \text{pd}_R(A),$$

and

$$\text{wd}_R(A) \leq \text{wd}_{R[\theta]}(A) \leq 1 + \text{wd}_R(A).$$

Proof. Inasmuch as $(R[\theta])_R$ is free, any projective resolution for $A_{R[\theta]}$ is also a projective resolution for A_R, whence $\text{pd}_R(A) \leq \text{pd}_{R[\theta]}(A)$. As in the case of polynomials [6, Part III, Theorem 6], there exists an exact sequence $0 \to A \otimes_R R[\theta] \to A \otimes_R R[\theta] \to A \to 0$ of right $R[\theta]$-modules, from which we infer that $\text{pd}_{R[\theta]}(A) \leq 1 + \text{pd}_R(A)$.

The inequalities for weak dimension are proved in the same way.

Proposition 3. *If R is any nonzero differential ring, then*

$$1 \leq \text{r. gl. dim. } R[\theta] \leq 1 + \text{r. gl. dim. } R.$$

In case r. gl. dim. $R < \infty$, *then also*

$$\text{r. gl. dim. } R \leq \text{r. gl. dim. } R[\theta].$$

Proof. The inequality r. gl. dim. $R[\theta] \leq 1 +$ r. gl. dim. R is clear from Proposition 2. Observing that $R[\theta]/\theta R[\theta]$ cannot be isomorphic to a right ideal of $R[\theta]$, we see that r. gl. dim. $R[\theta] \geq 1$.

Now assume that r. gl. dim. $R = n < \infty$, and choose a right R-module A with $\text{pd}_R(A) = n$. There is an obvious exact sequence $0 \to A \to A \otimes_R R[\theta] \to B \to 0$, and certainly $\text{pd}_R(B) \leq n$, hence we infer from the long exact sequence for Ext that $\mathrm{pd}_R(A \otimes_R R[\theta]) = n$. In view of Proposition 2, it now follows that r. gl. dim. $R[\theta] \geq n$.

We now give an example of a differential ring R with r. gl. dim. $R = \infty$ and r. gl. dim. $R[\theta] = 1$.

Choose a field F of characteristic 2 and consider the derivation $\partial/\partial x$ on $F[x]$. The ideal (x^2) is a differential ideal because of characteristic 2, hence we obtain a differential ring $R = F[x]/(x^2)$. As an F-algebra, R has a basis $1, w$ such that $w^2 = 0$ and $\delta w = 1$. Observing an exact sequence $0 \to wR \to R \to wR \to 0$, we infer that gl. dim. $R = \infty$.

Given any right $R[\theta]$-module A, we may choose a basis $\{b_\alpha\}$ over F for the vector space $B = \{b \in A \mid bw = 0\}$. Set $a_\alpha = b_\alpha \theta$ for all α, and note that $a_\alpha w = b_\alpha$. It is routine to check that A_R is free with basis $\{a_\alpha\}$, hence we obtain $\mathrm{pd}_{R[\theta]}(A) \leq 1$ from Proposition 2, and thus r. gl. dim. $R[\theta] = 1$.

This example may be extended somewhat as follows. Set $R_1 = R$ as above, and for $n > 1$ set $R_n = R[x_1, \cdots, x_{n-1}]$ and extend δ to a derivation of R_n by defining $\delta x_i = 0$ for all i. This choice of δ ensures that the x_i all commute with θ in $R_n[\theta]$, whence $R_n[\theta] \cong R[\theta][x_1, \cdots, x_{n-1}]$ and so r. gl. dim. $R_n[\theta] = n$. We summarize these examples as follows:

For each positive integer n, there exists a commutative noetherian differential ring R_n *such that* gl. dim. $R_n = \infty$ *and* r. gl. dim. $R_n[\theta] = n$.

3. **Commutative noetherian differential rings.** For some of the results in this section, we require that R be an algebra over the rationals, in which case we follow Kaplansky [5] in using the term *Ritt algebra* to denote a commutative differential ring which is an algebra over the rationals.

Lemma 4. *Let R be a noetherian Ritt algebra. If A is any nonzero finitely generated right $R[\theta]$-module, then there exists a chain of $R[\theta]$-submodules $A_0 = 0 < A_1 < \cdots < A_k = A$ and a collection of prime ideals $P_1, \cdots, P_k$ of R such that for each i, either $A_i/A_{i-1} \cong R[\theta]/P_iR[\theta]$ or else P_i is a differential ideal and A_i/A_{i-1} is a torsion-free (R/P_i)-module.*

Proof. As observed in [1, p. 68], $R[\theta]$ is right noetherian, and so A has ACC on $R[\theta]$-submodules. Thus it suffices to show that there exists a nonzero $R[\theta]$-submodule B of A and a prime ideal P in R satisfying one of the two relationships described.

Choose a nonzero $x \in A$ whose R-annihilator $P = \{r \in R \mid xr = 0\}$ is maximal among the R-annihilators of nonzero elements of A, and set $B = xR[\theta]$. According to [7, Theorem 6], P is a prime ideal of R. If P is a differential ideal, then $R[\theta]P = PR[\theta]$ and we obtain $BP = 0$. In this case the maximality of P now ensures that $B_{R/P}$ is torsion-free. Clearly $B \cong R[\theta]/J$ for some right ideal J of $R[\theta]$ such that $PR[\theta] \subseteq J$ and $J \cap R = P$. If P is not a differential ideal, then an easy argument due to Hart in [4, Lemma 2.4] shows that $J = PR[\theta]$.

Theorem 5. *Let R be a noetherian Ritt algebra with* gl. dim. $R = n < \infty$. *If* $\mathrm{pd}_R(R/P) < n$ *for all prime differential ideals P of R, then* r. gl. dim. $R[\theta] = n$.

Proof. If $n = 0$, then R must contain a prime ideal P which is generated by an idempotent e. Differentiating the identities $e^2 = e$ and $(1 - e)^2 = 1 - e$, we find that $\delta e = 0$ and so P is a differential ideal. But $\mathrm{pd}_R(R/P) < 0$ is impossible, hence we must have $n \neq 0$.

We have r. gl. dim. $R[\theta] \geq n$ by Proposition 3, hence it suffices to prove that $\mathrm{pd}_{R[\theta]}(A) \leq n$ for any nonzero finitely generated $A_{R[\theta]}$. Since $R[\theta]$ is right noetherian, it is enough to show that $\mathrm{wd}_{R[\theta]}(A) \leq n$. In view of Lemma 4, we may also assume that there is a prime ideal P in R such that either $A \cong R[\theta]/PR[\theta]$ or else P is a differential ideal and A is a torsion-free (R/P)-module.

If $A \cong R[\theta]/PR[\theta]$, then obviously $\mathrm{wd}_{R[\theta]}(A) \leq \mathrm{wd}_R(R/P) \leq n$.

Now assume that P is a differential ideal and that A is a torsion-free (R/P)-module. Then A_R can be embedded in some direct product T of copies of the quotient field of R/P, and since R/P is noetherian we see that $T_{R/P}$ is flat. We are given $\mathrm{pd}_R(R/P) \leq n - 1$, and we infer from [2, Proposition 4.1.2, p. 117] that $\mathrm{wd}_R(T) \leq n - 1$ also. There is an exact sequence $0 \to A \to T \to C \to 0$, and certainly $\mathrm{wd}_R(C) \leq n$, hence it follows from the long exact sequence for Tor that $\mathrm{wd}_R(A) \leq n - 1$. By Proposition 2, $\mathrm{wd}_{R[\theta]}(A) \leq n$.

Lemma 6. *Let R be a commutative noetherian differential ring with* gl. dim. $R = n < \infty$. *If R is local with maximal ideal M, and if M is a differential ideal, then* r. gl. dim. $R[\theta] = n + 1$.

Proof. We already have r. gl. dim. $R[\theta] \leq n + 1$ by Proposition 3. According to [6, Part III, Theorems 12, 13], R has classical Krull dimension n, and by [8, (h)] R must also have Krull dimension n in the sense of [8].

Inasmuch as M is a differential ideal, R/M is a differential ring and $MR[\theta]$ is a two-sided ideal of $R[\theta]$ with $R[\theta]/MR[\theta] \cong (R/M)[\theta]$. Observing that the right Krull dimension of $(R/M)[\theta]$ is at least 1, we see from [1, Lemma 2.c.1] that the right Krull dimension of $R[\theta]$ is at least $n + 1$.

Now $R[\theta]$ is a filtered noetherian ring whose associated graded ring is the commutative regular noetherian ring $R[x]$. According to a theorem of Roos in [11] (quoted in [1, p. 78]), the right Krull dimension of $R[\theta]$ is at most r. gl. dim. $R[\theta]$, hence we obtain r. gl. dim. $R[\theta] \geq n + 1$.

Lemma 7. *Let R be a commutative differential ring, S any multiplicative subset of R.*

(a) *δ induces a derivation on R_S by the rule $\delta(r/s) = [(\delta r)s - r(\delta s)]/s^2$.*

(b) *The natural map ϕ: $R[\theta] \to R_S[\theta]$ makes $R_S[\theta]$ into a flat left $R[\theta]$-module such that the multiplication map $R_S[\theta] \otimes_{R[\theta]} R_S[\theta] \to R_S[\theta]$ is an isomorphism.*

(c) r. gl. dim. $R_S[\theta] \leq$ r. gl. dim. $R[\theta]$.

Proof. (a) is an easy verification.

(b) We check that $\ker \phi = \{x \in R[\theta] \mid xs = 0$ for some $s \in S\}$, that each element of $\phi(S)$ is invertible in $R_S[\theta]$, and that every element of $R_S[\theta]$ may be written in the form $(\phi x)(\phi s)^{-1}$ for suitable $x \in R[\theta]$, $s \in S$. Thus $R_S[\theta]$ is just the classical localization of $R[\theta]$ with respect to the multiplicative set S, and the required properties follow as in the commutative case.

(c) According to (b), $R_S[\theta]$ is a right localization of $R[\theta]$ in the sense of [12], hence this inequality follows from [12, Corollary 1.3].

Theorem 8. *Let R be a commutative noetherian differential ring with* gl. dim. $R = n < \infty$. *If R contains a prime differential ideal P such that* $\mathrm{pd}_R(R/P) = n$, *then* r. gl. dim. $R[\theta] = n + 1$.

Proof. We already have r. gl. dim. $R[\theta] \leq n + 1$ by Proposition 3. According to [6, Part III, Theorem 11], gl. dim. $R_P \leq n$. There is a natural exact sequence $0 \to R/P \to R_P/PR_P \to A \to 0$, and we have $\mathrm{wd}_R(R/P) = n$, $\mathrm{wd}_R(A) \leq n$, hence it follows from the long exact sequence for Tor that $\mathrm{wd}_R(R_P/PR_P) = n$. Because $(R_P)_R$ is flat, the weak dimension of R_P/PR_P over R_P must be at least n, and we conclude that gl. dim. $R_P = n$.

Now R_P is a differential ring as in Lemma 7, and PR_P is a differential ideal of R_P because P is a differential ideal of R, hence we obtain r. gl. dim. $R_P[\theta] = n + 1$ from Lemma 6. Therefore r. gl. dim. $R[\theta] \geq n + 1$, by Lemma 7.

Corollary 9. *Let R be a commutative noetherian differential ring with* gl. dim. $R = n < \infty$, *and assume that R has prime characteristic p. If R contains a prime ideal Q such that $\delta^p(Q) \subseteq Q$ and* $\mathrm{pd}_R(R/Q) = n$, *then* r. gl. dim. $R[\theta] = n + 1$.

Proof. Because of characteristic p, it follows from Leibnitz' rule that δ^p is a derivation on R. Likewise, we obtain $\theta^p a = a\theta^p + \delta^p a$ for all $a \in R$, whence the set $R[\theta^p]$ is a subring of $R[\theta]$ isomorphic to $(R, \delta^p)[\theta]$. Since Q is a prime differential ideal in (R, δ^p), Theorem 8 says that r. gl. dim. $R[\theta^p] = n + 1$. Observing that $R[\theta]$ is a free right and left $R[\theta^p]$-module, we obtain r. gl. dim. $R[\theta] \geq n + 1$ from [1, Lemma 2.b.4], and by Proposition 3 we are done.

Combining Theorems 5 and 8 immediately yields the following formula:

Theorem 10. *Let R be a noetherian Ritt algebra with* gl. dim $R = n < \infty$. *If* $k = \sup\{\mathrm{pd}_R(R/P) \mid P$ *is a prime differential ideal of* $R\}$, *then* r. gl. dim. $R[\theta] = \max\{n, k + 1\}$.

4. **Applications and remarks.** (a) The *Weyl algebra* $A_1(S)$ over a ring S is just $S[x][\theta]$, where we use the derivation $\delta = \partial/\partial x$ on $S[x]$. If S is a field of characteristic 0, then r. gl. dim. $A_1(S) = 1$, while if S is a field of positive characteristic, then r. gl. dim. $A_1(S) = 2$ [9, Theorem, p. 345]. We generalize these results as follows:

Theorem 11. *Let S be a commutative noetherian ring with* gl. dim. $S = n < \infty$. *If S is an algebra over the rationals, then* r. gl. dim. $A_1(S) = n + 1$, *while if S has positive characteristic, then* r. gl. dim. $A_1(S) = n + 2$.

Proof. Let R denote the ring $S[x]$ with the derivation $\partial/\partial x$.

Assuming that S is an algebra over the rationals, let P be any differential ideal of R, and set $J = P \cap S$. It is easy to see that $P = JR[x]$, whence $\mathrm{pd}_R(R/P) \leq \mathrm{pd}_S(S/J) \leq n$. According to Theorem 5, r. gl. dim. $A_1(S) = n + 1$.

Now assume that S has positive characteristic. In view of [6, Part III, Theorems 11, 13], S is a regular noetherian ring, hence [7, Theorem 168] says that S is a finite direct product of domains. Thus we need only consider the case when S is a domain, and here the characteristic of S must be a prime p.

In view of [7, Exercise 7, p. 53], we must have $\mathrm{pd}_S(S/P) = n$ for some prime ideal P of S. Let $Q = P + xR$, which is a prime ideal of R such that $R/Q \cong S/P$. Now $\mathrm{pd}_R(R/Q) = n + 1$ by [6, Part III, Theorem 3], and it is clear that $\delta^p(Q) \subseteq Q$, hence Corollary 9 says that r. gl. dim. $A_1(S) = n + 2$.

(b) For any ring S, one can define a ring $F_1(S) = S[[x]][\theta]$ using the derivation $\partial/\partial x$ on $S[[x]]$. Proceeding as in Theorem 11, we obtain the following results, of which the first part is a special case of [1, Theorem 4.2].

Theorem 12. *Let S be a commutative noetherian ring with* gl. dim. $S =$

$n < \infty$. *If* S *is an algebra over the rationals, then* r. gl. dim. $F_1(S) = n + 1$, *while if* S *has positive characteristic, then* r. gl. dim. $F_1(S) = n + 2$.

(c) [4, Theorem 2.6] says that if R is a commutative noetherian integral domain (not a field) with finite Krull dimension, and if δ is a derivation on R such that $R[\theta]$ is a simple ring, then r. gl. dim. $R[\theta]$ = gl. dim. R. As noted in [4, Lemma 2.4], it follows from these hypotheses that R is a Ritt algebra and that the only differential ideals in R are 0 and R. Thus for the case gl. dim. $R < \infty$, this theorem is a consequence of Theorem 5. For the case gl. dim. $R = \infty$, however, the proof of this theorem relies on the inequality gl. dim. $R \leq$ r. gl. dim. $R[\theta]$ and so is open to question.

(d) [3, Theorem 2] is another theorem which relies on the inequality gl. dim. $R \leq$ r. gl. dim. $R[\theta]$ in the infinite-dimensional case and so is open to question. In the finite-dimensional case, this result is an easy consequence of Theorem 8.

(e) We conclude by noting that Theorems 5 and 10 may fail in cases of positive characteristic. For example, let F be any algebraically closed field of positive characteristic, and set $R = F[x]$, $\delta = \partial/\partial x$. Since the maximal ideals of R are generated by linear polynomials, none of them can be differential ideals. Thus we have $\mathrm{pd}_R(R/P) = 0 <$ gl. dim. R for all prime differential ideals P of R, whereas Theorem 11 says that

$$\text{r. gl. dim. } R[\theta] = 2 > \text{gl. dim. } R.$$

Perhaps Theorems 5 and 10 may be made to hold in general by using more differential ideals P than just the prime ones.

Added in proof.

(f) Some results related to those in this paper have appeared in S. M. Bhatwadekar, *On the global dimension of Ore-extensions*, Nagoya Math. J. 50 (1973), 217–225. Bhatwadekar's Theorem 2.3 gives the rational case of our Theorem 11, and his Theorem 1.1 can be obtained as a direct consequence of our Theorem 5.

(g) Other results related to the present paper appear in G. S. Rinehart and A. Rosenberg, *The global dimension of Ore extensions and Weyl algebras*, Communications in Algebra (to appear). Their Theorem 2.6 generalizes our Theorem 11 to the case of Weyl algebras of arbitrary degree.

(h) In a sequel to the present paper, we generalize the results given here to the case of a commutative noetherian ring with a finite collection of commuting derivations. Also, we answer the question raised in Remark (e): Theorem 10 holds for commutative noetherian differential rings R with gl. dim. $R < \infty$, provided k is set equal to the supremum of $\mathrm{pd}_R(R/P)$ where P ranges over all *primary* differential ideals of R.

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