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ESSENTIAL MAPS AND EMBEDDINGS OF ANNULI IN NONORIENTABLE M^3

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ABSTRACT. In this note we give an example which shows that the existence of an essential map of an annulus in a nonorientable 3-manifold does not guarantee the existence of an essential embedding in that manifold.

I. **Introduction.** It has been reported by F. Waldhausen that the existence of an "essential" map of an annulus into a compact orientable 3-manifold guarantees the existence of an essential embedding of an annulus in that 3-manifold. It is natural to ask if the result above is true for nonorientable 3-manifolds. In this note we give an example to show that the answer to the question above is in general no.

Feustel has proved [2] that the existence of a certain type of essential map of an annulus into a compact (not necessarily orientable) 3-manifold guarantees the existence of an embedding having the same properties. Our example also shows that Feustel's conditions are necessary in case the 3-manifold is not orientable.

II. **Notation.** We shall let A denote an annulus, F a mobius band, S^1 the 1-sphere, and M a 3-manifold. All spaces will be simplicial complexes and all maps will be piecewise linear. We shall denote the boundary of a manifold N by ∂N and the components of ∂A by c_1 and c_2. We let α be an arc embedded in A such that $c_j \cap \alpha$ is an endpoint of α for $j = 1, 2$. Such an arc is a *spanning arc* of A. A map $f: (A, \partial A) \to (M, \partial M)$ is *essential* if

(1) $f_*: \pi_1(A) \to \pi_1(M)$ is monic,

(2) the arc $f(\alpha)$ is not homotopic rel its boundary to an arc in ∂M.

We remark that condition (2) above is independent of our choice of the

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spanning arc α. A connected surface S embedded in a manifold M is *incompressible in* M if S is not the 2-sphere and the natural map $i_*: \pi_1(S) \to \pi_1(M)$ induced by inclusion is monic.

We shall say that a simple loop λ_1 in the interior of $F \times [0, 1]$ is *knotted in* $F \times [0, 1]$ if the complement of the interior of a regular neighborhood R of λ_1 in $F \times [0, 1]$ is not homeomorphic to the product to a klein bottle with $[0, 1]$. Let λ be a 1-sided simple loop in the interior of F and t a point in $[0, 1]$. One can find a knotted loop λ_1 in $F \times [0, 1]$ by altering the loop $\lambda \times \{t\}$ inside a regular neighborhood of one of the points on $\lambda \times \{t\}$.

Let F_1 and F_2 be surfaces in M or ∂M. Then F_1 is *parallel* to F_2 if and only if there exists an embedding of $F_1 \times [0, 1]$ in M such that $F_1 = F_1 \times \{0\}$, and F_2 is the closure of $\partial(F_1 \times [0, 1]) - F_1 \times \{0\}$.

III. **Construction of the manifold** M. Let x be a point in S^1 and $M_1 = F \times S^1$. Let R be a regular neighborhood of $\lambda \times \{x\}$ in M_1. Then R is homeomorphic to $F \times [0, 1]$. Let $\mathfrak{D}$ be a disk embedded in R such that $\mathfrak{D} \cap \partial R = \partial \mathfrak{D}$ and $R - \mathfrak{D}$ is connected. Let λ_1 be a knotted loop in R such that λ_1 meets $\mathfrak{D}$ in a single point and crosses $\mathfrak{D}$ at that point. Let R_1 be a regular neighborhood of λ_1 in R. Denote ∂R_1 by K_1 and ∂R by K. Let M_2^1 be the 3-manifold obtained by removing the interior of R_1 from M_1. Let $h: M_2^1 \to M_2^2$ be a homeomorphism and M the union of M_2^1 and M_2^2 with the identification $y = h(y)$ for $y \in K_1$.

IV. **There is an essential map of** A **into** M. Let $p: (A, \partial A) \to (F, \partial F)$ be a covering map. Let x_0 be a point in S^1 such that R does not meet $F \times \{x_0\} \subset M_1$. Then we may define $f: (A, \partial A) \to (M_2^1, \partial M_2^1)$ by $f(s) = (p(s), x_0)$. We claim that f is essential. Clearly condition (1) is satisfied since the natural map $i_*: \pi_1(M_2^1) \to \pi_1(M)$ induced by inclusion is monic.

Let $(\hat{M}, \hat{p})$ be the orientable double cover of M. Now there is a map $\hat{f}: A \to \hat{M}$ such that $\hat{p}\hat{f} = f$. It can be seen that $\hat{f}(c_1)$ and $\hat{f}(c_2)$ lie on different components of $\partial \hat{M}$. It follows that $f(\alpha)$ is not homotopic rel its boundary to an arc in ∂M since such a homotopy could be lifted to $\hat{M}$.

V. **There are no essential embeddings of** A **in** M. Suppose that $g: (A, \partial A) \to (M, \partial M)$ is an essential embedding. We claim that we may assume that $g(A)$ does not meet K_1. After a general position argument, we may assume that $g^{-1}(K_1)$ is a collection (possibly empty) of simple closed loops. Since K_1 and $g(A)$ are incompressible in M, any loop in $K_1 \cap g(A)$, nullhomotopic in M, is nullhomotopic both on $g(A)$ and on K_1. It follows from

standard arguments that we may suppose that no loop in $K_1 \cap g(A)$ is null-homotopic in M. Similarly we may suppose that $K \cap g(A)$ and $h(K) \cap g(A)$ are collections of simple loops and that no loop in either collection is null-homotopic in M. We suppose that $g(A)$ meets K_1.

Note that up to isotopy, the only simple essential loop on $T = \partial M_2^1 \cap \partial M$ which is homologous to a loop on K is $\partial F \times \{x_0\}$. We may assume that $g(A)$ meets T and that there is an annulus $A_1 \subset g(A)$ such that $A_1 \cap (T \cup K) = \partial A_1$. Thus $\partial A_1 \cap T$ is twice the generator of $\pi_1(R)$. Let $A_2 \subset g(A)$ be an annulus such that $A_2 \cap (K \cup K_1) = \partial A_2$ and A_2 meets both K_1 and K. Since $\partial A_1 \cap \partial A_2 \cap K$ is either empty or all of $\partial A_1 \cap K$ and ∂A_1 is orientable, standard arguments will show that λ_1 must have been unknotted. It follows that we may assume that $g(A)$ does not meet K_1 and that $g(\partial A) \subset \partial M_2^1$.

It is now clear that we may regard g as an essential embedding of A in M_2^1 or simply $M_1 - \lambda_1$. Let $(\hat{M}, \hat{p})$ be the orientable double cover of M_1. Since $g(A) \subset M_1 - \lambda_1$ and $[g(c_1)] \in \hat{p}_* \pi_1(\hat{M}_1)$, we can find a map $\hat{g}: A \to \hat{M}_1$ such that $\hat{p}\hat{g} = g$ and $\hat{g}(A) \subset \hat{M}_1 - \hat{p}^{-1}\lambda_1$. Suppose that $\hat{g}(\partial A)$ lies on a single component of $\partial \hat{M}_1$. Let $\rho: \hat{M}_1 \to \hat{M}_1$ be the nontrivial covering translation of $\hat{M}_1$. Since $\hat{M}_1$ is homeomorphic to the product of a torus T with $[0, 1]$, it will follow from [3, Corollary 3.2] that $\hat{g}(A)$ and $\rho\hat{g}(A)$ are parallel to annuli in $\partial \hat{M}_1$. Since $\hat{p}^{-1}\lambda_1$ is connected and $\hat{g}(\partial A)$ and $\rho\hat{g}(\partial A)$ lie on distinct components of $\partial \hat{M}_1$, it will follow that either $\hat{g}(\alpha)$ or $\rho\hat{g}(\alpha)$ is homotopic rel its boundary to an arc in $\partial \hat{M}_1$ in the complement of $\hat{p}^{-1}\lambda_1$. One can project this homotopy to show that $g(A)$ was not essential.

Suppose that $\hat{g}$ carries c_1 and c_2 to distinct components of $\partial \hat{M}_1$. We observe that $\hat{g}(A) \cap \rho\hat{g}(A)$ is empty since g is an embedding. Let N_1 and N_2 be the closures of the components of the complement of $\hat{M}_1 - (\hat{g}(A) \cup \rho\hat{g}(A))$. Then we may suppose that $\hat{p}^{-1}\lambda_1 \subset N_1$ since $\hat{p}^{-1}(\lambda_1)$ is connected. We claim that $\rho N_1 = N_1$. Suppose that there is a point $z \in (N_1 - \rho N_1)$. Then $\rho z \in N_2$ and we can find a path β in N_1 from z to λ_1 which does not meet $\hat{g}(A) \cup \rho\hat{g}(A)$. But now $\rho\beta$ is a path which does not meet $\hat{g}(A) \cup \rho\hat{g}(A)$. It follows that $\rho\beta \subset N_2$ and $\hat{p}^{-1}(\lambda_1)$ meets N_2. This is impossible. Since $\rho(N_1 - \rho N_1) = \rho N_1 - N_1$, $N_1 = \rho N_1$. Similarly $\rho N_2 = N_2$.

It is now clear that $g(A)$ separates M_1 into two components whose closures are $X_1 = pN_1$ and $X_2 = pN_2$. We obtain the commutative diagram shown in Figure 1 where all homomorphisms are induced by inclusion. Since $\pi_1(M_1) = z \oplus z$, $\pi_1(M_1)$ is not a nontrivial free product with amalgamation. Thus i_1 or i_2 must be onto. Since $\lambda_1 \subset X_1$, i_2 is onto. It follows from

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
\pi_1(X_1) \arrow[dr, "i_3"'] & & \pi_2(X_2) \arrow[dl, "i_4"] \\
 & \pi_1(M_1) &
\end{tikzcd}$$

Figure 1

[1, Theorem (1.1)] that X_2 is homeomorphic to $A \times [0, 1]$ and that $g(\alpha)$ is homotopic in X_2 rel its boundary to an arc in ∂M_1.

Thus g was not essential.

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