ESSENTIAL MAPS AND EMBEDDINGS
OF ANNULI IN NONORIENTABLE $M^3$

R. J. DAIGLE AND C. D. FEUSTEL

ABSTRACT. In this note we give an example which shows that the existence of an essential map of an annulus in a nonorientable 3-manifold does not guarantee the existence of an essential embedding in that manifold.

I. Introduction. It has been reported by F. Waldhausen that the existence of an "essential" map of an annulus into a compact orientable 3-manifold guarantees the existence of an essential embedding of an annulus in that 3-manifold. It is natural to ask if the result above is true for nonorientable 3-manifolds. In this note we give an example to show that the answer to the question above is in general no.

Feustel has proved [2] that the existence of a certain type of essential map of an annulus into a compact (not necessarily orientable) 3-manifold guarantees the existence of an embedding having the same properties. Our example also shows that Feustel's conditions are necessary in case the 3-manifold is not orientable.

II. Notation. We shall let $A$ denote an annulus, $F$ a mobius band, $S^1$ the 1-sphere, and $M$ a 3-manifold. All spaces will be simplicial complexes and all maps will be piecewise linear. We shall denote the boundary of a manifold $N$ by $\partial N$ and the components of $\partial A$ by $c_1$ and $c_2$. We let $\alpha$ be an arc embedded in $A$ such that $c_j \cap \alpha$ is an endpoint of $\alpha$ for $j = 1, 2$. Such an arc is a spanning arc of $A$. A map $f: (A, \partial A) \to (M, \partial M)$ is essential if

1. $f_*: \pi_1(A) \to \pi_1(M)$ is monic,
2. the arc $f(\alpha)$ is not homotopic rel its boundary to an arc in $\partial M$.

We remark that condition (2) above is independent of our choice of the
spanning arc \(a\). A connected surface \(S\) embedded in a manifold \(M\) is \textit{incompressible in} \(M\) if \(S\) is not the 2-sphere and the natural map \(i_*: \pi_1(S) \to \pi_1(M)\) induced by inclusion is monic.

We shall say that a simple loop \(\lambda_1\) in the interior of \(F \times [0, 1]\) is \textit{knotted in} \(F \times [0, 1]\) if the complement of the interior of a regular neighborhood \(R\) of \(\lambda_1\) in \(F \times [0, 1]\) is not homeomorphic to the product to a klein bottle with \([0, 1]\). Let \(\lambda\) be a 1-sided simple loop in the interior of \(F\) and \(t\) a point in \([0, 1]\). One can find a knotted loop \(\lambda_1\) in \(F \times [0, 1]\) by altering the loop \(\lambda \times \{t\}\) inside a regular neighborhood of one of the points on \(\lambda \times \{t\}\).

Let \(F_1\) and \(F_2\) be surfaces in \(M\) or \(\partial M\). Then \(F_1\) is \textit{parallel to} \(F_2\) if and only if there exists an embedding of \(F_1 \times [0, 1]\) in \(M\) such that \(F_1 = F_1 \times \{0\}\), and \(F_2\) is the closure of \(\partial(F_1 \times [0, 1]) - F_1 \times \{0\}\).

**III. Construction of the manifold \(M\).** Let \(x\) be a point in \(S^1\) and \(M_1 = F \times S^1\). Let \(R\) be a regular neighborhood of \(\lambda \times \{x\}\) in \(M_1\). Then \(R\) is homeomorphic to \(F \times [0, 1]\). Let \(\mathcal{D}\) be a disk embedded in \(R\) such that \(\mathcal{D} \cap \partial R = \partial \mathcal{D}\) and \(R - \mathcal{D}\) is connected. Let \(\lambda_1\) be a knotted loop in \(R\) such that \(\lambda_1\) meets \(\mathcal{D}\) in a single point and crosses \(\mathcal{D}\) at that point. Let \(R_1\) be a regular neighborhood of \(\lambda_1\) in \(R\). Denote \(\partial R_1\) by \(K_1\) and \(\partial R\) by \(K\). Let \(M_2^1\) be the 3-manifold obtained by removing the interior of \(R_1\) from \(M_1\). Let \(h: M_2^1 \to M_2^2\) be a homeomorphism and \(M\) the union of \(M_2^1\) and \(M_2^2\) with the identification \(y = h(y)\) for \(y \in K_1\).

**IV. There is an essential map of \(A\) into \(M\).** Let \(p: (A, \partial A) \to (F, \partial F)\) be a covering map. Let \(x_0\) be a point in \(S^1\) such that \(R\) does not meet \(F \times \{x_0\} \subset M_1\). Then we may define \(f: (A, \partial A) \to (M_2^1, \partial M_2^1)\) by \(f(s) = (p(s), x_0)\). We claim that \(f\) is essential. Clearly condition (1) is satisfied since the natural map \(i_*: \pi_1(M_2^1) \to \pi_1(M)\) induced by inclusion is monic.

Let \((\hat{M}, \hat{p})\) be the orientable double cover of \(M\). Now there is a map \(\hat{f}: A \to \hat{M}\) such that \(\hat{p} \hat{f} = f\). It can be seen that \(\hat{f}(c_1)\) and \(\hat{f}(c_2)\) lie on different components of \(\partial \hat{M}\). It follows that \(f(a)\) is not homotopic rel its boundary to an arc in \(\partial M\) since such a homotopy could be lifted to \(\hat{M}\).

**V. There are no essential embeddings of \(A\) in \(M\).** Suppose that \(g: (A, \partial A) \to (M, \partial M)\) is an essential embedding. We claim that we may assume that \(g(A)\) does not meet \(K_1\). After a general position argument, we may assume that \(g^{-1}(K_1)\) is a collection (possibly empty) of simple closed loops. Since \(K_1\) and \(g(A)\) are incompressible in \(M\), any loop in \(K_1 \cap g(A)\), null-homotopic in \(M\), is nullhomotopic both on \(g(A)\) and on \(K_1\). It follows from
standard arguments that we may suppose that no loop in $K_1 \cap g(A)$ is null-homotopic in $M$. Similarly we may suppose that $K \cap g(A)$ and $\partial K \cap g(A)$ are collections of simple loops and that no loop in either collection is null-homotopic in $M$. We suppose that $g(A)$ meets $K_1$.

Note that up to isotopy, the only simple essential loop on $T = \partial M^1_2 \cap \partial M$ which is homologous to a loop on $K$ is $xF \times \{x_0\}$. We may assume that $g(A)$ meets $T$ and that there is an annulus $A_1 \subset g(A)$ such that $A_1 \cap (T \cup K) = \partial A_1$. Thus $\partial A_1 \cap T$ is twice the generator of $\pi_1(T)$. Let $A_2 \subset g(A)$ be an annulus such that $A_2 \cap (K \cup K_1) = \partial A_2$ and $A_2$ meets both $K_1$ and $K$. Since $\partial A_1 \cap \partial A_2 \cap K$ is either empty or all of $\partial A_1 \cap K$ and $\partial A_1$ is orientable, standard arguments will show that $\lambda_1$ must have been unknotted. It follows that we may assume that $g(A)$ does not meet $K_1$ and that $g(\partial A) \subset \partial M_2$.

It is now clear that we may regard $g$ as an essential embedding of $A$ in $M_1$ or simply $M_1 - \lambda_1$. Let $(\tilde{M}, \tilde{F})$ be the orientable double cover of $M_1$. Since $g(A) \subset M_1 - \lambda_1$ and $[g(c_1)] \in \pi_1(\tilde{M}_1)$, we can find a map $\tilde{g}: A \to \tilde{M}_1$ such that $\tilde{g}^* = g$ and $\tilde{g}(A) \subset \tilde{M}_1 - \tilde{F}^{-1}\lambda_1$. Suppose that $\tilde{g}(\partial A)$ lies on a single component of $\partial \tilde{M}_1$. Let $\rho: \tilde{M}_1 \to \tilde{M}_1$ be the nontrivial covering translation of $\tilde{M}_1$. Since $\tilde{M}_1$ is homeomorphic to the product of a torus $T$ with $[0, 1]$, it will follow from [3, Corollary 3.2] that $\tilde{g}(A)$ and $\rho g(A)$ are parallel to annuli in $\partial \tilde{M}_1$. Since $\tilde{F}^{-1}\lambda_1$ is connected and $\tilde{g}(\partial A)$ and $\rho g(\partial A)$ lie on distinct components of $\partial \tilde{M}_1$, it will follow that either $\tilde{g}(\lambda)$ or $\rho g(\lambda)$ is homotopic rel its boundary to an arc in $\partial \tilde{M}_1$ in the complement of $\tilde{F}^{-1}\lambda_1$. One can project this homotopy to show that $g(A)$ was not essential.

Suppose that $\tilde{g}$ carries $c_1$ and $c_2$ to distinct components of $\partial \tilde{M}_1$. We observe that $\tilde{g}(A) \cap \rho \tilde{g}(A)$ is empty since $g$ is an embedding. Let $N_1$ and $N_2$ be the closures of the components of the complement of $\tilde{M}_1 - (\tilde{g}(A) \cup \rho \tilde{g}(A))$. Then we may suppose that $\tilde{F}^{-1}\lambda_1 \subset N_1$ since $\tilde{F}^{-1}(\lambda_1)$ is connected. We claim that $\rho N_1 = N_1$. Suppose that there is a point $z \in (N_1 - \rho N_1)$. Then $\rho z \in N_2$ and we can find a path $\beta$ in $N_1$ from $z$ to $\lambda_1$ which does not meet $g(A) \cup \rho g(A)$. But now $\rho \beta$ is a path which does not meet $\tilde{g}(A) \cup \rho g(A)$. It follows that $\rho \beta \subset N_2$ and $\tilde{F}^{-1}(\lambda_1)$ meets $N_2$. This is impossible. Since $\rho(N_1 - \rho N_1) = \rho N_1 - N_1$, $N_1 = \rho N_1$. Similarly $\rho N_2 = N_2$.

It is now clear that $g(A)$ separates $M_1$ into two components whose closures are $X_1 = \rho N_1$ and $X_2 = \rho N_2$. We obtain the commutative diagram shown in Figure 1 where all homomorphisms are induced by inclusion. Since $\pi_1(M_1) = z \Theta z$, $\pi_1(M_1)$ is not a nontrivial free product with amalgamation. Thus $i_1$ or $i_2$ must be onto. Since $\lambda_1 \subset X_1$, $i_2$ is onto. It follows from
[1, Theorem (1.1)] that $X_2$ is homeomorphic to $A \times [0, 1]$ and that $g(\alpha)$ is homotopic in $X_2$ rel its boundary to an arc in $\partial M_1$.

Thus $g$ was not essential.

REFERENCES


3. F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. (2) 87 (1968), 56–88. MR 36 #7146.

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061