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POINTWISE BOUNDS ON EIGENFUNCTIONS AND WAVE PACKETS IN N-BODY QUANTUM SYSTEMS. II

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ABSTRACT. We provide a simple proof (and mild improvement) of Schnol's result that L^2 eigenfunctions of $-\Delta + V$ are $O(\exp(-ar))$ for any $a > 0$ whenever $V \to \infty$ as $r \to \infty$.

Despite a rather large literature (reviewed in [6]) on the exponential falloff of Schrödinger operators, $-\Delta + V$, one of the strongest results is one of the first, that of Schnol [9], who asserts that L^2 solutions of $(-\Delta + V)\psi = E\psi$ obey pointwise bounds of the form

$$|\psi(r)| \leq C_a \exp(-ar) \tag{1}$$

if V is continuous and bounded below and E is in the discrete spectrum of $-\Delta + V$. The constant a in Schnol's result can be taken arbitrarily obeying $a < f(d(E))$, where f is a universal function depending on E and V only through the lower bound on V and $d(E)$ is the distance of E from the essential spectrum of $-\Delta + V$.

For the general multiparticle quantum system, Schnol's results have two obvious weaknesses: V is not bounded below in atomic and other systems of interest, and secondly Schnol's function $f(E)$ behaves as $\ln E$ as $E \to \infty$ instead of as $\sqrt{E}$ which is suggested by spherically symmetric examples and the theory of ordinary differential equations [8]. Much of the literature on the subject deals with these weaknesses. Due to the recent work of O'Connor [6], Combes-Thomas [2] and Simon [10], we now have nearly maximally good results for the case $V(r) = \sum V_{ij}(r_i - r_j)$ with $V_{ij}(x) \to 0$ as $x \to \infty$.

There is another case which is better handled by Schnol's result, namely where $V \to \infty$ at ∞ with V bounded below (generalized harmonic oscillator) and our goal in this note is to use the methods of the recent papers just quoted to obtain Schnol's results for this case.

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First, we can exploit the Combes-Thomas idea:

Theorem 1. *Let* $V = V_1 + V_2$ *where* $V_2 \geq 0$, $V_2 \in L^1_{loc}(\mathbf{R}^n)$ *and where* V_1 *is a form bounded perturbation of* $-\Delta$ *with form bound, a, less than* 1 *(e.g.* $V_1 \in L^{n/2}(\mathbf{R}^n)$ *will do if* $n \geq 3$*; see e.g.* [7]*). Let* $H = -\Delta + V$ *defined as a sum of quadratic forms* [4], [11]*. Suppose that* $-(1-a)\Delta + V_2$ *has compact resolvent (e.g. if* $\inf_{|r|>R} V_2(r) \to \infty$ *as* $R \to \infty$*). Then any* L^2 *eigenfunction of* $-\Delta + V$ *lies in the domain of* $\exp(cr)$ *for any* $c > 0$.

Proof. Let $H(\vec{b})$ be defined as $(i\vec{\nabla} - \vec{b})^2 + V$ for any $\vec{b} \in \mathbf{C}^n$. It is easy to see that $H(\vec{b})$ is an entire analytic family of type (B) in the sense of Kato [4] with invariant form domain $D(-\Delta^{1/2}) \cap Q(V_2)$. Moreover, since $H(0)$ has compact resolvent by hypothesis and $H(\vec{b})$ is unitarily equivalent to $H(0)$ if $\vec{b} \in \mathbf{R}^n$, $H(\vec{b})$ has compact resolvent for all $\vec{b}$. By mimicking the Combes-Thomas arguments, one easily sees that eigenvectors of H are entire vectors [5] for the group $\exp(i\vec{b} \cdot \vec{r})$ which implies our result. □

To obtain pointwise bounds on eigenfunctions we are *not* able to mimic Simon [10], who combines L^2 exponential bounds with L^∞ bounds of Kato [3] for vectors in $C^\infty(H)$, because Kato's methods only work for potentials going to zero at infinity [3], [10]. Instead we combine the L^2 bounds with L^∞ bounds of Davies [1] for analytic vectors of H. Use of Davies' ideas restricts us to V's which are bounded below, thereby recovering Schnol's result:

Theorem 2. *Let* $V \geq 0$, $V \in L^1_{loc}(\mathbf{R}^n)$, $\inf_{|r|>R} V(r) \to \infty$ *as* $R \to \infty$. *Let* $\psi \in L^2(\mathbf{R}^n)$ *lie in* $Q(-\Delta) \cap Q(V)$ *and obey* $-\Delta\psi + V\psi = E\psi$*. Then for any* $a > 0$*, there is a* C *with* $|\psi(r)| \leq C\exp(-a|r|)$.

Proof. It is obviously sufficient to show that for each $b \in \mathbf{R}^n$, $\phi \equiv \exp(b \cdot r)\psi \in L^\infty$. But ϕ obeys $H(ib)\phi = E\phi$ with $\phi \in L^2$ by Theorem 1. Now, write $H(ib) = H_0(ib) + V$ and consider the semigroup $\exp(-tH_0(ib))$. It is easy to write down an explicit kernel for it and see that:

(a) $\exp(-tH_0(ib))$ is positivity preserving;

(b) $\exp(-tH_0(ib))$: $L^2(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$ for $t > 0$.

Write $V_n = \min(V, n)$. Then $H_0(ib) + V_n$ converges to $H(ib)$ in strong resolvent sense as $n \to \infty$ so that

$$\exp(-tH(ib))\phi = \underset{n\to\infty}{\text{s-lim}} \left(\underset{m\to\infty}{\text{s-lim}} \left(\exp(-tH_0(ib)/m)\exp(-tV_n/m)\right)^m \right)\phi$$

and thus since $e^{-tV_n/m} \leq 1$ $(V \geq 0!)$ and (a)

$$|\exp(-tH(ib))\phi| \leq \exp(-tH_0(ib))|\phi|$$

pointwise. Thus, by (b), $\exp(-tH(ib))\phi = \exp(-tE)\phi$ is in L^∞ so that ϕ is in L^∞. □

Remark. Depending on how fast V goes to infinity we expect ψ to obey $\exp(-x^\alpha)$ bounds for $\alpha > 1$. We hope to return to this question in a future publication.

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