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P-SETS IN F'-SPACES

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ABSTRACT. A P-set is one which is interior to any zero set which contains it. An F'-space may be characterized as one in which the closure of a cozero set is a P-set. We study applications of P-sets to the topology of F'-spaces, and certain set-theoretical operations under which the class of P-sets is stable. A. I. Veksler has shown that in a basically disconnected space the closure of an arbitrary union of P-sets is a P-set, while in F'-spaces we are only able to prove this for countable unions. Our main result is an example of a set in the compact F-space $\beta N \setminus N$ which is not a P-set, but which is the closure of a union of P-sets. The set is related to the almost-convergent functions of G. G. Lorentz.

1. **Introduction.** A *P-set* in a topological space is a closed set which is interior to any zero set which contains it. Apparently the first explicit occurrence of infinite P-sets in the literature is in the work of V. T. Dikanova [D], where it is assumed that X is compact and extremally disconnected. Properties of P-sets in basically disconnected spaces were further developed by A. I. Veksler [V]. For both authors, P-sets arose naturally from the study of certain vector lattices, and their representation by function spaces.

The first significant occurrence of infinite P-sets in the American literature is the result of Henriksen and Isbell [H–I] that the 'support set' in $\beta N \setminus N$ of a nonnegative regular matrix is an infinite P-set. (See also [H–S, Theorem 4.1] and [A_1, Theorem 2.1].) In [A_2] it is shown that, under the continuum hypothesis, each such support set contains a family of 2^c nowhere dense P-sets, each the support of a regular Borel measure.

In §2 we show how P-sets may be used to characterize F'-spaces, and discuss some set-theoretic operations under which the class of P-sets (in an F'-space) is stable—for instance the closure of a countable union of P-sets is a P-set. In §3 we discuss P-sets in basically and extremally dis-

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connected spaces, and use P-sets to give an easy proof that F'-spaces with the countable chain condition are extremally disconnected.

A. I. Veksler [V, Theorem 2(b)] shows that in a compact basically disconnected space, the closure of an arbitrary union of P-sets is a P-set, while in compact F'-spaces we are only able to prove this for countable unions. In §4 we give a counterexample to show that the countability restriction is needed. Let $A \subset N$ be such that its characteristic function is almost-convergent in the sense of G. G. Lorentz (see [L] or [R_1]), let A^* be its cluster points in $\beta N \setminus N$, and let $K = \bigcap \{A^* : \chi_A \text{ almost conv.}\}$. A result of R. A. Raimi implies K is not a P-set. We show here that K is the closure of an uncountable union of support sets of regular matrices, each of which is a P-set by the Henriksen-Isbell theorem.

2. **P-sets in F'-spaces.** X will always be a completely regular Hausdorff space. We recall that a zero set has the form $f^{-1}(0)$ for some $f \in C(X)$, and a cozero set is the complement of a zero set. By complete regularity, cozero sets are a system of basic neighborhoods. An F'-space is one in which disjoint cozero sets have disjoint closures ([G–H] or [C–H–N]). A closed set is called a P-set if it is interior to any zero set which contains it. Equivalently, a closed set is a P-set if it is disjoint from the closure of any cozero set in its complement.

2.1. **Theorem.** *The following are equivalent*: (a) *X is an F'-space*, (b) *the closure of every cozero set is a P-set.*

Proof. (a) *implies* (b). Let $K = \mathrm{cl}\, A$, where A is cozero (and $\mathrm{cl}\, A$ means 'closure of A'), and suppose B is a cozero set disjoint from K. Then $\mathrm{cl}\, B$ is disjoint from K by definition of F'-space. Hence K is a P-set.

(b) *implies* (a). Let A and B be disjoint cozero sets. Then $\mathrm{cl}\, A \cap B = \varnothing$, where $\mathrm{cl}\, A$ is, by assumption, a P-set, and B is cozero. Hence $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \varnothing$, and so X is an F'-space.

2.2. **Theorem.** *Let X be an F'-space.*

(a) *If K_n are compact P-sets, then $\mathrm{cl} \bigcup_n K_n$ is a P-set.*

(b) *If Ω is the first uncountable ordinal and $\{K_\alpha : \alpha < \Omega\}$ are compact P-sets such that $\alpha < \beta$ implies $K_\alpha \supset K_\beta$, then $K = \bigcap \{K_\alpha : \alpha < \Omega\}$ is a P-set.*

(c) *If X is compact and each closed set has a neighborhood system of cardinality at most $\aleph_1$, then the finite intersection of P-sets is a P-set.*

Proof. (a) Let Z be a zero set with $\mathrm{cl} \bigcup K_n \subset Z$. Each K_m is a P-set, so for each m there exists open V_m with $K_m \subset V_m \subset Z$. Since K_m is com-

pact and X is completely regular, there exists a cozero set W_m with $K_m \subset W_m \subset V_m \subset Z$, and hence $\mathrm{cl} \bigcup K_n \subset \mathrm{cl} \bigcup W_n \subset Z$. But $\bigcup W_n$ is a cozero set. By 2.1, $\mathrm{cl} \bigcup W_n$ is a P-set, and hence is interior to Z. Thus $\mathrm{cl} \bigcup K_n$ is interior to Z, and hence is a P-set.

(b) Let Z be a zero set with $K \subset Z$. Then $Z = \bigcap_n A(n)$, where each $A(n)$ is open. For each n there exists $\alpha(n) < \Omega$ with $K_{\alpha(n)} \subset A(n)$. (For otherwise $\{K_\alpha \setminus A(n): \alpha < \Omega\}$ would be a family of compact sets with f.i.p., and hence would have nonvoid intersection, contrary to $K \subset A(n)$.) Since Ω is the first uncountable ordinal, there exists α with $\alpha(n) < \alpha < \Omega$ for all n, whence $K_\alpha \subset K_{\alpha(n)} \subset A(n)$ for all n. Now $K_\alpha \subset Z$, and since K_α is a P-set, we have $K \subset K_\alpha \subset$ interior Z. Hence K is a P-set.

(c) We need two lemmas.

2.3. **Lemma.** *Let X satisfy the hypotheses of* 2.2(c), *and let K be closed in X. The following are equivalent*: (a) *K is a P-set*, (b) *K can be written $K = \bigcap\{A_\alpha: \alpha < \Omega\}$, where A_α is cozero, and $\alpha < \beta$ implies $A_\alpha \supset \mathrm{cl}\, A_\beta$.*

Proof. (b) *implies* (a). This follows from 2.2(b). (a) *implies* (b). Let $\{B_\alpha: \alpha < \Omega\}$ be a well-ordering of open neighborhoods of K. By normality, there is a cozero set A_1 with $K \subset A_1 \subset B_1$. Suppose $\beta < \Omega$ and we have $\{A_\alpha: \alpha < \beta\}$ such that $A_\alpha \subset B_\alpha$, and $\alpha < \gamma < \beta$ implies $A_\alpha \supset \mathrm{cl}\, A_\gamma \supset A_\gamma \supset K$. By normality of X, $Z = \bigcap\{A_\alpha: \alpha < \beta\}$ is a zero set with $K \subset Z$, and since K is a P-set there exists open W with $K \subset W \subset Z$. By normality there exists a cozero set A_β with $K \subset A_\beta \subset \mathrm{cl}\, A_\beta \subset W \cap B_\beta \subset Z$.

2.4. **Lemma.** *If A and B are cozero sets in F'-space X, then* $\mathrm{cl}(A \cap B) = \mathrm{cl}\, A \cap \mathrm{cl}\, B$.

Proof. We adapt an argument from [G–J, p. 85]. Clearly, $\mathrm{cl}(A \cap B) \subset \mathrm{cl}\, A \cap \mathrm{cl}\, B$. To prove the reverse inclusion, let $p \in \mathrm{cl}\, A \cap \mathrm{cl}\, B$ and V be a cozero set neighborhood of p. Then $p \in \mathrm{cl}(A \cap V)$ and $p \in \mathrm{cl}(B \cap V)$. Since $A \cap V$ and $B \cap V$ are cozero sets and X is an F'-space, we have $A \cap B \cap V \neq \varnothing$. Since V is an arbitrary cozero neighborhood of p, we have $p \in \mathrm{cl}(A \cap B)$.

2.5. **Proof of** 2.2(c). Let K and L be P-sets. By Lemma 2.3 we write $K = \bigcap\{\mathrm{cl}\, A_\alpha: \alpha < \Omega\}$ and $L = \{\mathrm{cl}\, B_\beta: \beta < \Omega\}$. Using Lemma 2.4,

$$K \cap L = \bigcap_{\alpha<\Omega} \bigcap_{\beta<\Omega} \mathrm{cl}(A_\alpha \cap B_\beta) = \bigcap_{\alpha<\Omega} K_\alpha.$$

By 2.2(b) each $K_\alpha = \bigcap_{\beta<\Omega} \mathrm{cl}(A_\alpha \cap B_\beta)$ is a P-set, and again by 2.2(b) $K \cap L$ is a P-set. The general case of n sets follows by induction.

2.6. **Remarks.** (a) 2.2(c) fails for sequences of sets. In fact if K_n are distinct compact sets with $K_n \supset K_{n+1}$, then their intersection K is not a P-set. For choose $x_n \in K_n \setminus K_{n+1}$. Then $\mathrm{cl}\{x_n\}$ intersects K, while it is easy to see that a P-set is disjoint from the closure of any countable subset of its complement. (b) It has been shown anonymously that 2.2(c) can be proved by means of functional analysis in such a way that the cardinality assumptions may be discarded.

3. **Basically and extremally disconnected spaces.** Theorems 3.2 and 3.3 are due to A. I. Veksler. The proofs we give are direct and set-theoretic, while his proofs are based on vector lattice properties of the space of extended real functions on X. Theorem 3.4 was proved for compact totally disconnected F-spaces by K. Hoffman and A. Ramsay [H–R, Lemma 3]. The proof of our more general result is quite different from theirs.

3.1. **Lemma.** *Let X be an F'-space. If A is a cozero set and K a P-set, then* $\mathrm{cl}_X A \cap K = \mathrm{cl}_K(A \cap K)$.

Proof. Clearly, $\mathrm{cl}_X A \cap K \supset \mathrm{cl}_K(A \cap K)$. Suppose $p \notin \mathrm{cl}_K(A \cap K)$. We show $p \notin \mathrm{cl}_X A \cap K$. Now there is a cozero set B with $p \in B$ and $\varnothing = (B \cap K) \cap (A \cap K) = A \cap B \cap K$. But $A \cap B$ is a cozero set and K is a P-set, so using Lemma 2.4 we have $\varnothing = \mathrm{cl}_X(A \cap B) \cap K = \mathrm{cl}_X A \cap \mathrm{cl}_X B \cap K$. But $p \in B$, and hence $p \notin \mathrm{cl}_X A \cap K$.

3.2. **Theorem** [V, Theorem 3]. *Let X be basically disconnected (i.e., the closure of every cozero set is open), and K a compact P-set in X. Then K is basically disconnected in its subspace topology.*

Proof. Let A be a cozero set in K. Since K is compact, every continuous function on K can be extended to a continuous function on all X. Hence $A = K \cap B$, where B is cozero in X. By 3.1, $\mathrm{cl}_K A = \mathrm{cl}_K(K \cap B) = \mathrm{cl}_X B \cap K$. But $\mathrm{cl}_X B$ is open and closed in X, so $\mathrm{cl}_K A$ is open and closed in the induced topology of K.

3.3. **Theorem** [V, Theorem 2(b)]. *Let X be basically disconnected. Then the closure of an arbitrary union of P-sets is a P-set. Hence the class of P-sets is a complete lattice.*

Proof. Let $\{K_a : a \in A\}$ be P-sets, and $\mathrm{cl} \bigcup_a K_a \subset Z$, where Z is a zero set. For each $a \in A$, $K_a \subset \mathrm{int}\, Z$. But $\mathrm{int}\, Z$ is closed, so $\mathrm{cl} \bigcup_a K_a \subset \mathrm{int}\, Z$. Hence $\mathrm{cl} \bigcup_a K_a$ is a P-set.

3.4. **Theorem.** *An F'-space which satisfies the countable chain condition is extremally disconnected (i.e., the closure of an open set is always open).*

Proof. As is well known, a basically disconnected space which satisfies c.c.c. is extremally disconnected, so it suffices to prove X is basically disconnected. Let K be the closure of a cozero set in X, so that (by Theorem 2.1) K is a P-set. Let $\{A_n : n \in N\}$ be a maximal disjoint collection of cozero sets in $X \setminus K$. Then $A = \bigcup_n A_n$ is a cozero set dense in $X \setminus K$. It follows from the definition of P-set that $K \cap \operatorname{cl} A = \varnothing$, so that $\operatorname{cl} A = X \setminus K$. Thus K is open as well as closed.

3.5. **Corollary** (G. L. Seever [S, Theorem 2.2]). *If S is the carrier set of a positive Borel measure in a compact F-space, then S is extremally disconnected in its subspace topology.*

Proof. S is an F-space in its induced topology [S, p. 271], and it clearly satisfies c.c.c.

3.6. **Remark.** Professor W. W. Comfort has suggested the following alternative proof of Theorem 3.4. A space is called 'weakly Lindelöf' if every open cover admits a countable subfamily with dense union. In [C–H–N, pp. 494–495] it is shown that an F'-space in which each open set is weakly Lindelöf is extremally disconnected, and so it suffices to show that under our hypotheses each open set is weakly Lindelöf. But given an open cover of open V, there exists a maximal disjoint collection of open sets, each contained in some element of the cover. The disjoint collection is countable with dense union, and yields a subfamily of the original cover which is countable with dense union.

4. **The example.** The example given in Theorem 4.2 will show that Veksler's theorem (Theorem 3.3 above), which holds for basically disconnected spaces, fails for the compact F-space $\beta N \setminus N$.

4.1. **Preliminaries.** $f \in C^*(N)$ is *almost-convergent* (a-c) to t if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(j+k) = t \tag{4.1a}$$

uniformly in k. G. G. Lorentz [L, Theorem 1] proved that f is almost convergent to t iff $\phi(f) = t$ for every invariant mean ϕ on $C^*(N)$, i.e., every translation-invariant nonnegative linear functional such that $\phi(e) = 1$, where e is the unit function.

If $f \in C^*(N)$, let f' be its extension to βN and f^* the restriction of f'

to $\beta N \backslash N$. If $A \subset N$, let A' be its closure in βN and $A^* = A' \cap (\beta N \backslash N)$. Then $N^* = \beta N \backslash N$. As noted in [$R_2$], each invariant mean on $C^*(N)$, i.e., on $C(\beta N)$, may be represented by a Borel probability measure with support in N^* which is invariant under the extension to βN of the map $n \to n+1$ on N. Then f is a-c to t iff $\int f^* dm = t$ for each measure m representing an invariant mean. If ϕ is an invariant mean, let K_ϕ be the support set of the measure representing ϕ, and let

$$K = \mathrm{cl} \bigcup \{K_\phi : \phi \text{ an invariant mean}\}. \tag{4.1b}$$

By [R_2, Theorem 2.4] we have

$$K = \bigcap \{A^* : A \subset N,\ \chi_A \text{ is a-c to } 1\}. \tag{4.1c}$$

Finally if $T = (t_{mn})$ is a nonnegative regular matrix, we define the *support set* L of T by

$$L = \bigcap \{A^* : A \subset N,\ T\text{-}\lim \chi_A = 1\}, \tag{4.1d}$$

where $T\text{-}\lim \chi_A = \lim_m \sum_n t_{mn} \chi_A(n)$. (This is equivalent to Definition 1.1 of [A_1].) The *bounded convergence field* of T is the set of $f \in C^*(N)$ such that $T\text{-}\lim(f)$ exists.

4.2. **Theorem.** *The set K is not a P-set, but there exists a family of P-sets $\{K_A : A \in S\}$, where S is uncountable, such that $K = \mathrm{cl} \bigcup \{K_A : A \in S\}$.*

Proof. We first show that K is not a P-set. In [R_1, pp. 711–712], Raimi produces a nonnegative $f \in C^*(N)$ such that (i) f is almost convergent to 0, and (ii) there does not exist a partition $N = A \cup B$ such that χ_A is almost convergent to 0 and $\lim(n \in B) f(n) = 0$. By (i), $\int f^* dm = 0$ whenever m is a measure on N^* representing an invariant mean. Since f is nonnegative, this implies that $f^* = 0$ on the support set of m, and hence, by formula (4.1b), $f^* = 0$ on K. We show that f^* does not vanish on any neighborhood of K so that K is not a P-set. Let B^* be an open and closed neighborhood of K, where $B \subset N$. Then obviously χ_B is almost-convergent to 1, and condition (ii) implies that $\limsup(n \in B) f(n) > 0$. Thus f^* does not vanish on B^*.

Now to produce the sets K_A. Each K_A will be the support of a regular translative matrix, i.e., a regular matrix $A = (a_{mn})$ such that

$$\lim (m \to \infty) \sum_n |a_{m,n} - a_{m,n+1}| = 0.$$

According to Lorentz [L], A is translative iff a-c $\subset C_A$, and $A\text{-}\lim f = \rho(f)$ for each $f \in$ a-c and invariant mean ρ. Let S be the set of all nonnegative

regular translative matrices. If $A \in S$, let K_A be the support set of A (as defined in 4.1).

(i) If $A \in S$, then $K_A \subset K$. To prove this, let $W \subset N$ be such that $K \subset W^*$. Then $\chi_W \in$ a-c $\subset C_A$, so A-lim $\chi_W = \rho(\chi_W) = 1$ for each invariant mean ρ. Hence $K_A \subset W^*$, and it follows that $K_A \subset K$.

(ii) $K = \mathrm{cl} \bigcup \{K_A : A \in S\}$. To prove this, let K_0 stand for the right-hand side of the equation. If the result is false, then there exists $x \in K \setminus K_0$. By complete regularity, there exists $f \in C^*(N)$ such that $0 \leq f^* \leq 1$, $f^*(x) > 0$, and $f^* = 0$ on K_0. This implies that A-lim $f = 0$ for all $A \in S$, while $\rho(f) > 0$ for some invariant mean ρ. (This last follows from formula (4.1b), and the fact that $f^* \geq 0$.) But this contradicts the following lemma of J. P. Duran [Du, Lemma 1]:

'Let x be a bounded sequence. Then there is an $A \in S$ and a $B \in S$ such that

$$\sup\{\rho(x) : \rho \text{ an invariant mean}\} = \lim_n (Ax)_n,$$

$$\inf\{\rho(x) : \rho \text{ an invariant mean}\} = \lim_n (Bx)_n.'$$

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REFERENCES

[A_1] R. E. Atalla, *On the multiplicative behavior of regular matrices*, Proc. Amer. Math. Soc. 26 (1970), 437–446. MR 42 #6633.

[A_2] ———, *Regular matrices and P-sets in $\beta N \setminus N$*, Proc. Amer. Math. Soc. 37 (1973), 157–162.

[C–H–N] W. W. Comfort, N. Hindman and S. Negrepontis, *F'-spaces and their products with P-spaces*, Pacific J. Math. 28 (1969), 489–502. MR 39 #3440.

[D] V. T. Dikanova, *Boundedness conditions on sets in extended K-space*, Sibirsk. Mat. Ž. 9 (1968), 804–815 = Siberian Math. J. 9 (1968), 602–609. MR 38 #4952.

[Du] J. P. Duran, *Strongly regular matrices, almost convergence, and Banach limits*, Duke Math. J. 39 (1972), 497–502.

[G–H] L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. 82 (1956), 366–391. MR 18, 9.

[G–J] L. Gillman and M. Jerison, *Rings of continuous functions*, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.

[H–I] M. Henriksen and J. Isbell, *Multiplicative summability methods and the Stone-Čech compactification*. II, Notices Amer. Math. Soc. 11 (1964), 90–91. Abstract #608–116.

[H–S] J. D. Hill and W. T. Sledd, *Approximation in bounded summability fields*, Canad. J. Math. 20 (1968), 410–415. MR 36 #5561.

[H–R] K. Hoffman and A. Ramsay, *Algebras of bounded sequences*, Pacific J. Math. 15 (1965), 1239–1248.

[L] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948), 167–190. MR 10, 367.

[R_1] R. A. Raimi, *Convergence, density, and τ-density of bounded sequences*, Proc. Amer. Math. Soc. 14 (1963), 708–712. MR 27 #3966.

[R_2] ———, *Homeomorphisms and invariant measures for $\beta N \setminus N$*, Duke Math. J. 33 (1966), 1–12. MR 33 #6608.

[S] G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. 133 (1968), 267–280. MR 37 #1976.

[V] A. I. Veksler, *P-sets in topological spaces*, Dokl. Akad. Nauk SSSR 193 (1970), 510–513 = Soviet Math. Dokl. 11 (1970), 953–956. MR 43 #5480.

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