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ASYMPTOTIC DISTRIBUTION OF NORMALIZED ARITHMETICAL FUNCTIONS

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ABSTRACT. Let $f(n)$ be an arbitrary arithmetical function and let A_N and B_N be sequences of real numbers with $0 < B_N \to +\infty$ with N. We give a sufficient condition for $(f(n) - A_N)/B_N$ to have a limiting distribution. The case when $f(n)$ is defined by $f(n) = \Sigma g(d)$, where the summation is over all divisors d of n and $g(d)$ is any given arithmetical function, is discussed in more detail. A concrete example is given as an application of our result, in which example $f(n)$ is neither additive nor multiplicative. Our method of proof is to approximate $f(n)$ by a suitably chosen additive function, as proposed in [4], and then to apply general theorems available for additive functions.

1. The general theorem. Let $f(n)$ be an arbitrary arithmetical function and let A_N and B_N be sequences of real numbers with $0 < B_N \to +\infty$ as $N \to +\infty$. Let $N\nu_N(n: \cdots)$ denote the number of those integers n, not exceeding N, for which the property stated in the dotted space holds. Let finally $F(x)$ be a proper distribution function, that is, $F(x)$ is increasing, left continuous and its limits at $+\infty$ and $-\infty$ are one and zero, respectively. We say that the normalized arithmetical function $(f(n) - A_N)/B_N$ has the limiting distribution $F(x)$, if, as $N \to +\infty$,

$$\nu_N(n: f(n) - A_N < xB_N) = F(x) + o(1), \tag{1}$$

for all continuity points of $F(x)$. Our aim in the present paper is to determine the sequences A_N and B_N for which (1) holds. This will be achieved by choosing strongly additive functions $G_N(n)$ which are "close" to $f(n)$, a term to be made specific below, and for which the relation

$$\nu_N(n: G_N(n) - A_N^* < xB_N) = F(x) + o(1) \tag{2}$$

is known to hold with some sequences A_N^* and B_N. In (2) again, $N \to +\infty$

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and its validity is assumed for all continuity points of $F(x)$. We then deduce that (2) implies (1) with the same B_N and $F(x)$ and with A_N determined by A_N^* and by our concept of "closeness" of arithmetical functions. The details are as follows.

Put

$$d(f, G_N, a_N) = N^{-1} \sum_{n=1}^{N} (f(n) - G_N(n) - a_N)^2, \tag{3}$$

where a_N is a sequence of real numbers. Writing

$$f(n) - a_N - A_N^* = (f(n) - G_N(n) - a_N) + (G_N(n) - A_N^*),$$

the well-known Markov inequality implies that, if

$$d(f, G_N, a_N)/B_N^2 \to 0 \quad \text{as } N \to +\infty, \tag{4}$$

then (1) and (2) are equivalent if we put $A_N = a_N + A_N^*$. (Though this is a very frequently applied argument in the theory of distributions of arithmetical functions, the reader may want to refer to [7, p. 45] for details.) Since general solutions for (2) are known, our aim is to guarantee the validity of (4), and thus to make the value in (3) as small as possible. The strongly additive function $G_N(n)$, asymptotically minimizing (3), was determined by one of us in a recent paper [4]. For this strongly additive function, at prime numbers p,

$$G_N(p) = \{p^2/(p-1)\}N^{-1} \sum_{k=1}^{[N/p]} f(kp) - \{p/(p-1)\}N^{-1} \sum_{k=1}^{N} f(k). \tag{5}$$

We shall use (5) as a guide rather than definition for $G_N(n)$ and in applications the exact value of (5) will be replaced by asymptotic expressions. We point out that, when a specific choice of the asymptotic expression for (5) has been made, in most known cases the normalizing constants A_N^* and B_N in (2) are given by

$$A_N^* = \sum_{p \le N} \frac{G_N(p)}{p} \quad \text{and} \quad B_N^2 = \sum_{p \le N} \frac{G_N^2(p)}{p}. \tag{6}$$

We shall concentrate mainly on these choices of A_N^* and B_N. It may be of interest to remark here that in a recent paper [5] a solution for (2) was given with normalizing constants different from those in (6). The general result of [5] is, however, quite complicated and it would be an interesting work to make refinements of those results.

We now summarize the conclusions in the preceding arguments.

Theorem 1. *Let $f(n)$ be an arbitrary arithmetical function and $G_N(n)$ be a strongly additive function, determined by any asymptotic expression of the right-hand side of* (5). *Assume that* (2) *holds with the values given in* (6). *Then the validity of* (4) *implies* (1) *with* $A_N = a_N + A_N^*$.

Note that we have a freedom in the choice of a_N in (3). This can help in many cases to guarantee the validity of (4), which is our major assumption. Indeed, if for a choice of $G_N(n)$ and a_N,

$$d(f, G_N, a_N) \sim D_N^2 = O(B_N^2), \tag{7}$$

then

$$d(f, G_N, a_N + D_N) = d(f, G_N, a_N) + D_N^2 - \frac{2D_N}{N} \sum_{n=1}^{N} (f(n) - G_N(n) - a_N).$$

Therefore, if a_N is chosen so that

$$N^{-1} \sum_{n=1}^{N} (f(n) - G_N(n) - a_N) \sim D_N, \tag{8}$$

then (4) is automatically satisfied with $d(f, G_N, a_N + D_N)$. This freedom in choosing a_N will be exploited in the next section where we investigate a concrete class of functions for $f(n)$, covering the additive and multiplicative functions.

2. A special case. In this section we give a more specific form of Theorem 1 for the case when

$$f(n) = \sum_{d|n} g(d), \tag{9}$$

where $g(d)$ is any given arithmetical function. First of all, notice that if $g(d) = 0$ for all d except when d is a power of a prime number, then $f(n)$ is additive and, evidently, any additive function can be obtained in this way. On the other hand, if $g(d)$ is multiplicative, then so is $f(n)$, and, by the Moebius inversion formula, all multiplicative functions can be represented in the form of (9). However, our aim is to obtain a general theorem which does not make these restrictions on $g(d)$. We shall prove the following result.

Theorem 2. *Let $g(d)$ be an arithmetical function and define $f(n)$ by* (9). *Let $G_N(n)$ be strongly additive which is defined at primes p by the formula*

$$G_N(p) = p \sum_{d \le N;\, p|d} \frac{g(d)}{d} - \frac{p}{p-1} \sum_{d=[N/p]+1}^{N} \frac{g(d)}{d}. \tag{10}$$

Assume that (2) *holds with the constants in* (6) *and that* $B_N \to +\infty$ *with* N. *Then, if*

$$N^{-1}\sum_{n=1}^{N} f^2(n) - \left(\sum_{d-1}^{N}\frac{g(d)}{d}\right)^2 = O(B_N^2), \tag{11}$$

the normalized arithmetical function

$$\left(f(n) - \sum_{d=1}^{N}\frac{g(d)}{d} - D_N\right)\Big/B_N,$$

where D_N *is a suitable sequence of numbers satisfying* $D_N = O(B_N)$, *has a limiting distribution which coincides with the one obtained in* (2).

Before giving the proof, let us make two remarks. First of all, we wish to re-emphasize that our assumption about the validity of (2) is not a strong restriction, as very general solutions are known for (2) to hold (see, in particular, [7, p. 58]). Hence, in most cases, our only assumption is (11). Our second remark is to point out that Theorem 2 is a straight extension of the Erdös-Kac theorem, or the more general result of Kubilius [7, p. 58]. Indeed, as we have remarked, $f(n)$ in (9) reduces to additive functions if $g(d) = 0$ whenever d is not the power of a prime number, and it becomes strongly additive, if $g(d) \neq 0$ only if d is a prime number. Now, in the definition (10) of $G_N(p)$, the second term tends to zero, as $N \to +\infty$, for the class H of Kubilius, and the first sum contains only a single nonzero term, namely, when $d = p$. Thus, for all $p \leq N$, $G_N(p) = g(p) = f(p)$, which in turn yields that $G_N(n) = f(n)$ for all $n \leq N$.

We now turn to the proof of Theorem 2.

Proof. We shall apply Theorem 1, and therefore we have to show that the expressions in (5) and (10) are asymptotically equal and we have to specify a_N. For this goal, first observe that

$$\begin{aligned} N^{-1}\sum_{n=1}^{N} f(n) &= N^{-1}\sum_{n=1}^{N}\sum_{d|n} g(d) = N^{-1}\sum_{d=1}^{N} g(d)\left[\frac{N}{d}\right] \\ &= \sum_{d=1}^{N}\frac{g(d)}{d} + O\left(N^{-1}\sum_{d=1}^{N}|g(d)|\right), \end{aligned} \tag{12}$$

and that, as is well known for any strongly additive function,

$$N^{-1}\sum_{n=1}^{N} G_N(n) \sim \sum_{p\leq N}\frac{G_N(p)}{p}. \tag{13}$$

For giving an asymptotic formula in (5), notice that

$$f(kp) = \sum_{d|kp} g(d) = \sum_{d|k} g(d) + \sum{}' g(dp^t),$$

where Σ' is summation over $d|k$, $(d, p) = 1$, and t is defined by $p^{t-1}|k$ but $p^t \nmid k$. Thus

$$\begin{aligned} N^{-1} \sum_{k=1}^{[N/p]} f(kp) &= N^{-1} \sum_{k=1}^{[N/p]} f(k) + N^{-1} \sum_{k=1}^{[N/p]} \sum{}' g(dp^t) \\ &\sim \frac{1}{p} \sum_{d=1}^{[N/p]} \frac{g(d)}{d} + \left(1 - \frac{1}{p}\right) \sum_{d \le N;\, p|d} \frac{g(d)}{d}. \end{aligned} \tag{14}$$

(12) and (14) now yield that we can choose $G_N(p)$ by the formula (10) to approximate the expression of (5).

Turning to (4), we get

$$\begin{aligned} d(f, G_N, a_N) &= N^{-1} \sum_{n=1}^{N} f^2(n) + N^{-1} \sum_{n=1}^{N} G_N^2(n) + a_N^2 \\ &\quad - \frac{2}{N} \sum_{n=1}^{N} f(n) G_N(n) - \frac{2a_N}{N} \sum_{n=1}^{N} (f(n) - G_N(n)). \end{aligned} \tag{15}$$

As is well known, the strong additivity of $G_N(n)$ yields (see [7, (3.5), p. 34])

$$N^{-1} \sum_{n=1}^{N} G_N^2(n) = \left(\sum_{p \le N} \frac{G_N(p)}{p} \right)^2 + O\left(\sum_{p \le N} \frac{G_N^2(p)}{p} \right). \tag{16}$$

On the other hand,

$$\sum_{n=1}^{N} f(n) G_N(n) = \sum_{n=1}^{N} \left(\sum_{d|n} g(d) \right) \left(\sum_{p|n} G_N(p) \right) = \sum_{p \le N} G_N(p) \sum_{k=1}^{[N/p]} f(kp),$$

and thus, by the exact value (10) of $G_N(p)$ and by (14),

$$N^{-1} \sum_{n=1}^{N} f(n) G_N(n) = \sum_{p \le N} \frac{G_N(p)}{p} \sum_{d=1}^{N} \frac{g(d)}{d} + O\left(\sum_{p \le N} \frac{G_N^2(p)}{p} \right). \tag{17}$$

Bearing in mind (8), we choose

$$a_N \sim \sum_{d=1}^{N} \frac{g(d)}{d} - \sum_{p \le N} \frac{G_N(p)}{p}. \tag{18}$$

Thus (15), (16) and (17) yield

$$d(f, G_N, a_N) = N^{-1} \sum_{n=1}^{N} f^2(n) - \left(\sum_{d=1}^{N} \frac{g(d)}{d} \right)^2 + O(B_N^2),$$

and therefore, by the assumption (11),

$$d(f, G_N, a_N) = O(B_N^2).$$

But this is exactly the formula (7), from which we deduced that with a suitable sequence $D_N = O(B_N)$, $d(f, G_N, a_N + D_N)$ satisfies (4). Therefore Theorem 1 is applicable with a_N of (18) being replaced by $a_N + D_N$, where $D_N = O(B_N)$. As Theorem 1 says, the normalizing constants are

$$A_N = a_N + D_N + A_N^* \sim \sum_{d=1}^{N} \frac{g(d)}{d} + D_N$$

and B_N itself, where the notations (6) and (18) apply. Theorem 2 is thus established.

Throughout this paper, the approximating functions $G_N(n)$ can vary with N and they are actually defined for $n \leq N$ only. However, if $\Sigma g(d)/d$ converges, then the second term in (10) tends to zero, and, by letting $N \to +\infty$, $G_N(n)$ can be replaced by a single strongly additive function. If, in addition, the tail of $\Sigma g(d)/d$ tends to zero sufficiently fast, the main result of Galambos [4] becomes applicable, that is, a limiting distribution of $f(n)$, defined in (9), exists without any normalization. This case also reobtains some of the results in Erdös and Hall [2], and, at the same time, gives an extension of that investigation by not requiring that $g(d) > 0$. The general case, when $\Sigma g(d)/d$ converges, however, still remains open, since our assumption of $B_N \to +\infty$ with N excludes most of these cases from our present investigation.

We now give an example as an application of Theorem 2. Let $g(d) = 1$ if $d = p^\alpha q^\beta$ with primes p and q and $\alpha \geq 0$, $\beta \geq 0$, and let $g(d) = 0$ otherwise. Then for $f(n)$, defined in (9), we have from Theorem 2, that, with a well defined real number c,

$$\nu_N(n\colon f(n) - (\log\log N)^2 < (c + x)(\log\log N)^{3/2}) \to F(x),$$

where

$$F(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(-t^2/2)\, dt. \tag{19}$$

Indeed, by (10), for any fixed prime p (the asymptotic equation is in terms of $N \to +\infty$),

$$G_N(p) \sim p \sum{}' p^{-\alpha} q^{-\beta} = \sum{}' p^{-(\alpha-1)} q^{-\beta}$$
$$= (1 + o(1)) \sum_{\alpha \geq 1} p^{-(\alpha-1)} \sum_{\beta \geq 1;\, q \leq N} q^{-\beta} = \{p/(p-1)\}(1 + o(1)) \log\log N,$$

where Σ' denotes summation over primes q and integers $\alpha \geq 1$, $\beta \geq 0$ with $p^\alpha q^\beta \leq N$. In terms of N, we can take any asymptotic value of $G_N(p)$ above, thus $G_N(p) = p/(p-1)$ loglog N can be taken. Also, since we are interested in asymptotic distribution, we can equally take $G_N(p) =$ loglog N, and thus

$$G_N(n) = U(n) \log\log N, \tag{20}$$

where $U(n)$ is the number of prime divisors of n. Since by the Erdös-Kac theorem, for $G_N(n)$ in (20) and $F(x)$ of (19),

$$\nu_N(n\colon G_N(n) - (\log\log N)^2 < x(\log\log N)^{3/2}) \to F(x),$$

Theorem 2 implies our claim by showing that

$$D_N^2 = N^{-1} \sum_{n=1}^{N} (f(n) - U(n) \log\log N)^2 = c(\log\log N)^3(1 + o(1)).$$

This last asymptotic formula follows by elementary but somewhat complicated calculations, hence its details are omitted (the reader can find estimates of a similar nature in [6]).

We conclude with a remark. In our approach it was not essential that the argument n of $\nu f(n)$ should run through the consecutive integers. Whenever the asymptotic distribution of an additive function is known to exist on a sequence $m_1 < m_2 < \cdots$ of integers, our argument remains unchanged. For such extensions of the theory of asymptotic distribution of additive functions, see the survey (Galambos [3]).

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