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POSITIVE LINEAR OPERATORS CONTINUOUS FOR STRICT TOPOLOGIES

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ABSTRACT. If A is an SW algebra of real-valued functions on a set X equipped with the weak topology for A, and if A separates its zero sets, then many results valued for $C^b(X)$ equipped with a strict topology remain true when A is equipped with a strict topology. The concepts of α-additivity and tight positive linear operators are introduced. It is shown that if T is a positive linear map on A into z-separating SW algebra B and if $T(1_A) = 1_B$, then there exists a continuous function ϕ on Y (the domain of elements in B) into X such that $Tf(y) = f(\phi(y))$ if and only if T is an algebraic homomorphism and τ-additive.

Introduction. Let X be a completely regular Hausdorff space. An algebra A of real-valued functions on X is an SW algebra if A separates the points of X, contains the constants and is uniformly closed. If A is an SW algebra which separates its zero sets, then Kirk [4] has shown that A^* can be represented as the space $M(z(A))$ of finitely additive set functions on the Baire sets of X. Let X_A be the extreme points of positive face in the unit ball in A^* equipped with $\sigma(A^*, A)$. Let τ_A be the relative $\sigma(A^*, A)$ topology on X embedded into X_A. Then X_A is the compactification of (X, τ_A) for which $f \in C^b(X, \tau_A)$ has a continuous extension iff $f \in A$. Using Kirk's characterization and the techniques of Sentilles [9] many results on the strict topologies on $C^b(X)$ carry over to A.

If A and B are two z-separating SW algebras defined on X, Y respectively and are equipped with a strict topology β_α, then continuous positive linear maps can be characterized in terms of additivity conditions. Also if $C = \{T \in L(A, B): \theta \leq T, T(1_A) = 1_B\}$ then F. F. Bonsall, J. Lindenstrauss, and R. R. Phelps [1] have shown that the extreme points of C are the alge-

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braic homomorphisms, and recently J. Hoffman-Jørgensen [3] has shown that when $A = C^b(X)$ and $B = C^b(Y)$ then the algebraic homomorphisms which are β_0 continuous have a continuous map ϕ on Y into X such that $Tf(y) = f(\phi(y))$. We shall show in a different manner that such a ϕ exists for $T \in C$ iff T is an extreme point of C and τ-additive.

1. Basic definitions and preliminaries. An SW algebra A is algebraically, topologically, and lattice isomorphic to $C(X_A)$ under $f \to \overline{f}$ for $\overline{f}$ the unique continuous extension of $f \in A$ from X to X_A. Also A is z-separating if for Z_1, Z_2 disjoint zero sets in X, there is an $f \in A$ with $f(Z_1) = 0 \leq f \leq 1 = f(Z_2)$.

For α an infinite cardinal, we define

$$L_\alpha = \left\{ Q \subset X_A \setminus X \colon Q = \bigcap_{\gamma \in I} \overline{Z}_\gamma, \text{ card } I \leq \alpha, \ Z_\gamma \text{ is a zero set in } X \right\}.$$

Let τ be the least cardinal with the property that every closed set in X is the intersection of at most τ zero sets. Since the zero sets $z(A)$ of A form a basis for the τ_A-closed sets, $\tau \leq \text{card } 2^{X^\cdot}$ exists. For each $Q \in L_\alpha$, let β_Q be the Hausdorff locally convex topology on A generated by the seminorms $\{p_g \colon \overline{g}(Q) = 0,\ p_g(f) = \|f \cdot g\|,\ f, g \in A\}$. Let β_α denote the inductive limit topology on A of the B_Q topologies as Q varies over L_α. We shall use F.D. Sentilles' notation for β_0 as the strongest locally convex topology on A which agrees with the compact-open topology on norm bounded subsets of A. We note that $\beta_\alpha \geq \beta_{\alpha'}$ if $\alpha \leq \alpha'$, and $\beta_0 \leq \beta_\alpha$ for all α follows as in [9, 2.1].

Let A, B be z-separating SW algebras on X, Y respectively, then $T \in L^+(A, B)$ if T is a positive linear map on A into B.

Definitions. Let $T \in L^+(A, B)$ and $m \in M^+(z(A))$ and card $I \leq \alpha$, then

(1) T is *α-additive* if $Tf_\gamma \downarrow \theta$ whenever $f_\gamma \downarrow \theta$ in A for $\gamma \in I$.

(2) m is α-additive if $m(Z_\gamma) \to 0$ whenever $Z_\gamma \downarrow \emptyset$, $Z_\gamma \in z(A)$, $\gamma \in I$.

(3) T is *tight* if $\{Tf_i\}$ is norm bounded and uniformly convergent to θ on compact subsets of Y whenever $\{f_i\}$ has these properties in A.

The α-additive (respectively tight) elements of $M(z(A))$ generate a band denoted by M_α (respectively M_t). We remark that α-additivity will be extensively examined in a more general setting in [5].

2. Some results for $C^b(X)$ valid for A. We note that for $Q \in L_\alpha$, $C(X_A \setminus Q)$ is a Banach algebra with approximate identity $\{E_\lambda\}_{\lambda \in \Lambda}$ with $\Lambda \leq \alpha$ (see [4, 1.5]). Also as shown in [4, §6], $\mu \in M(z(A))$ has a unique extension $\overline{\mu}$ to the

Baire sets on X_A, which in turn has a unique extension to a Borel measure $\bar{\nu}$ on X_A. With these remarks and 5.5 of [4] we can essentially repeat the standard arguments to establish the results below.

1. **Proposition.** *$A(\beta_\alpha)$ and $A(\beta_0)$ are Hausdorff locally convex lattices.*

Proof. For β_0 the result is clear since the compact-open topology is solid. For β_α see [7, Theorem 2.2].

2. **Proposition.** *β_α, β_0 are the finest locally convex topologies agreeing with their respective selves on each rU for $r > 0$ and U the unit ball of A. The continuity of linear maps for each topology is determined on sets rU.*

Proof. Use the remarks above and the arguments given by Sentilles culminating with [9, 4.1].

3. **Proposition.** *β_α is the topology of uniform convergence on the weak* compact subsets of $M_\alpha^+(z(A))$.*

Proof. See the argument given by Mosiman in [7, 2.7].

4. **Proposition.** *The following are equivalent for $\theta \le \phi \in A(\|\cdot\|)^*$:*
(a) *ϕ is α-additive.*
(b) *μ is α-additive, where $\mu \in M(z(A))$ represents ϕ.*
(c) *$\bar{\mu}(Q) = 0$ for all $Q \in L_\alpha$.*
(d) *ϕ is β_α continuous.*

Proof. The proof uses the remarks above and follows the arguments given in [10, 24, p. 174], [6, 2.4], and [9, 4.3].

5. **Proposition.** (a) $A(\beta_0)^* = M_t$.

(b) *$S \subset M_t$ is β_0 equicontinuous iff S is tight (norm bounded and for $\epsilon > 0$ there is a compact $K \subset X$ with $|\mu|(X \setminus K) < \epsilon$).*

(c) *β_0 is the topology of uniform convergence on the tight subsets of M_t.*

Proof. The arguments are those used by Sentilles in 4.3 and 5.1 of [9] where μ is the compact-regular Borel extension of $\mu \in M_t$. For the existence of such a unique extension of μ see, for example, [5, 2.12].

6. **Proposition.** *Let δ_p be a point functional for $p \in X_A$. Then $p \in X$ iff $p \in M_\tau$ iff $p \in M_t$.*

Proof. If $p \in X$, then δ_p is tight so τ-additive. If p is not in X, then

$\{f \in A: \bar{f}(p) = 1\}$ forms a net $\{f_\gamma\}$ with $f_\gamma \downarrow \theta$ but $\delta_p(f_\gamma) = 1$ so $\delta_p \notin M_t$.

3. **Positive linear maps.** In this section we shall assume A, B are z-separating SW algebras on (X, τ_A) and (Y, τ_B) respectively with compactifications X_A and Y_B. We note that $T \in L^+(A, B)$ is norm continuous so that T^* is a weak* continuous linear map on $M(z(B))$ into $M(z(A))$.

7. **Proposition.** *Let* $T \in L^+(A, B)$ *and let* $\alpha \leq \alpha'$. *Then the following are equivalent.*

(a) T *is* β_α-$\beta_{\alpha'}$ *continuous.*

(b) $T^*(M_{\alpha'}(z(B))) \subset M_\alpha(z(A))$.

(c) T *is* α-*additive.*

Proof. (a) $\Rightarrow$ (b). This follows from 4.

(b) $\Leftrightarrow$ (c). Use 6 and $M_t \subset M_{\alpha'}$.

(b) $\Rightarrow$ (a). Let W be a $\beta_{\alpha'}$ neighborhood of θ in B. Then by 3, there is a weak* compact subset D of $M^+_\alpha(z(B))$ such that $D^\circ \subset W$. Since T^* is weak* continuous and positive, it follows from 3 that $(T^*D)^\circ = V$ is a β_α neighborhood of θ in A. Clearly $T(V) \subset W$.

8. **Proposition.** *The following are equivalent for* $T \in L^+(A, B)$:

(a) T *is tight.*

(b) T *is* β_0-β_0 *continuous.*

(c) T^* *maps tight subsets of* $M(z(B))$ *into tight subsets of* $M(z(A))$.

Proof. (a)$\Leftrightarrow$(b). Since the norm bounded nets which β_0 converge to θ in A are the norm bounded nets which converge uniformly to θ on compact sets, the equivalence of (a) and (b) follows from the last statement in 2.

(b)$\Leftrightarrow$(c). This follows from 5 since $M_t(z(B))$ and $M_t(z(A))$ are the dual spaces of $B(\beta_0)$ and $A(\beta_0)$ respectively and since the topology β_0 is the topology of uniform convergence on the tight subsets of M_t.

We remark that clearly T is β_0-β_α continuous $\Rightarrow$ T is β_0-β_0 continuous $\Rightarrow$ T is β_α-β_0 continuous. However neither of the converse directions holds. For example, the identity operator on $C^b(X)$ is β_0-β_0 continuous but not β_0-β_α continuous in any case where $M_t \neq M_\tau$ or when there exists a weak* compact subset of M^+_t which is not tight since in both cases $\beta_0 \neq \beta_\tau$ by 5.8 in [9]. Also if $\mu \in M^+_\tau \backslash M_t$ then $T(f) = \mu(f)1$ is β_α-β_0 continuous on $C^b(X)$ into $C^b(X)$ but it is not β_0 - β_0 continuous.

Let us denote $\{T \in L^+(A, B): T(1_A) = 1_B\}$ by C where 1_A and 1_B are the identity functions in A, B respectively.

9. **Proposition.** *Let* $T \in C$. *Then the following are equivalent.*

(1) *T is an extreme point of C.*

(2) *T is an algebraic homomorphism.*

(3) *T is a lattice homomorphism.*

(4) $T^*(Y_B) \subset X_A$.

Proof. Let $\overline{T}$ be the extension of T from $L^+(A, B)$ to $L^+(C(X_A), C(Y_B))$ given by $\overline{T}(\overline{f}) = \overline{T(f)}$. Since the extension of f to $\overline{f}$ from A to $C(X_A)$ and the extension g to $\overline{g}$ from B to $C(Y_B)$ are Banach lattice isomorphisms, we see that T has one of the properties (a)–(c) above if and only if $\overline{T}$ has the same properties. Hence, the result follows from Ellis [2, Theorem 1] since $T^*(Y_B) = \overline{T}^*(Y_B)$.

10. **Proposition.** *Let $T \in C$. Then T is an extreme point of C and τ-additive iff there exists a continuous function ϕ on (Y, τ_B) into (X, τ_A) such that $Tf(y) = f(\phi(y))$.*

Proof. Let T be an extreme point of C and let T be τ-additive. Then for any $y \in Y$, we have that $\delta_y \in Y_B$ is τ-additive by 6. Hence, $\delta_y \in M_\tau(z(B))$ so that $T^*\delta_y \in M_\tau(z(A))$ by 7. However, $T^*\delta_y \in X_A$ by 9 so that $T^*\delta_y = \delta_{x_y}$ for some $x_y \in X$ by 6. Since A separates the points of X, $\phi(y) = X_y$ is a well-defined function on Y into X which is clearly τ_A-τ_B continuous.

Conversely, let such a ϕ exist. Then since T^* is weak* continuous and $\{\delta_y: y \in Y\}$ is weak*-dense in Y_B, we see that $T^*(Y_B) \subset \overline{\{\delta_x: x \in X\}} = X_A$. Thus by 9, T is an extreme point of C. If $f_\gamma \downarrow \theta$ in A then $\{Tf_\gamma\}$ is decreasing in B since T is positive. Since $\lim Tf_\gamma(y) = \lim f(\phi(y)) = 0$, it follows that T is τ-additive.

From this we obtain the following result which extends Mosiman and Wheeler's result [8, 3.1].

11. **Corollary.** *If T is an extreme point of C, then the following are equivalent.*

(a) *T is tight.*

(b) *T is τ-additive.*

(c) $T^*\{\delta_y: y \in Y\} \subset \{\delta_x: x \in X\}$.

Also if T satisfies any of the above then T is α-additive.

Proof. (b) and (c) are equivalent by 10 and 7.

Since T is an extreme point of C, $T^*(Y_B) \subset X_A$; hence, (c) follows from (a) because $\{\delta_y: y \in Y\} = Y_B \cap M_t(z(B))$ and $\{\delta_x: x \in X\} = X_A \cap M_t(z(A))$ by 6.

Let $\{f_\gamma\}$ be a norm bounded net of functions in A which converges uniformly to θ on compact subsets of X. Let $\epsilon > 0$ and let K be a compact subset of Y. Then by the weak* continuity of T^* and (c), it is easily seen that $K' = \{x \in X: \delta_x = T^*\delta_y$ for some $y \in K\}$ is a compact subset of X. Since T is norm continuous, $\{Tf_\gamma\}$ is norm bounded in B, and γ_0 chosen such that $|f_\gamma(x)| < \epsilon$ for all $x \in K'$ when $\gamma \geq \gamma_0$ implies that $|Tf_\gamma(y)| < \epsilon$ for all $y \in K$ whenever $\gamma \geq \gamma_0$. Hence, T is tight so (c)$\Longrightarrow$(a).

Finally, if (a), (b) or (c) holds, then, as in 10, it is easily seen that T is α-additive.

We mention that T can be β_α-β_α continuous but not β-β continuous even if T is an extreme point of C. For example, let $X = Y = (0, \Omega)$ for Ω the first uncountable ordinal and $(0, \Omega)$ is equipped with its order topology. Let δ_0 be a σ-additive linear functional given by $\delta_0(f) = f(x_f)$ where $x_f \in (0, \Omega)$ satisfies $f(y) = f(x_f)$ for all $y \geq x_f$. Then δ_0 is not τ-additive. Thus, if we define T by $Tf = \delta_0(f)1$, then T is an extreme point of C since it is an algebraic homomorphism, but T is σ-additive and not τ-additive.

Finally we point out that we can determine when a continuous ϕ from Y to X can be extended to Y_B, which could be βY for example.

12. **Proposition.** *Let ϕ be a continuous function on Y into X. Then ϕ has a continuous extension to Y_B if and only if $T \in C$ defined by $Tf(y) = f(\phi(y))$ is β_τ-$\|\cdot\|$ continuous.*

Proof. Let T be β_τ-$\|\cdot\|$ continuous. Let $f_\gamma \downarrow \theta$ on X. Then if W is a β_τ-neighborhood of θ, there is a weak* compact subset $V \subset M_\tau^+$ such that $V^\circ \subset W$ by 3. Then $\langle f_\gamma, \phi\rangle$ converges uniformly to zero for all $\phi \in V$ by Dini's theorem, so there is a γ_0 such that $\gamma \geq \gamma_0$ implies $f_\gamma \in V^\circ \subset W$. Hence, $\{f_\gamma\}$ converges to θ for the β_τ topology. Thus, $\{Tf_\gamma\}$ converges to θ uniformly on Y. If $\delta_p \in Y_B$, then $\{\delta_p(Tf_\gamma)\}$ converges to zero, so it follows that $T^*\delta_p$ is τ-additive. Consequently, $T^*(Y_B) \subset \{\delta_x: x \in X\}$ and we extend ϕ continuously to Y_B by $\phi(p) = x$ for $T^*(\delta_p) = \delta_x$.

Conversely let T be defined by $Tf(y) = f(\phi(y))$ where ϕ has a continuous extension from Y to Y_B. Let $\{f_\gamma\}$ β_τ-converge to θ. Then we may assume $\{f_\gamma\}$ is norm bounded by 2. Since $T^*(Y_B) = \{\delta_x: x = \phi(p)$ for $p \in Y_B\}$, and $T^*(Y_B)$ is compact in X_A, $\{f_\gamma\}$ converges uniformly to θ on $T^*(Y_B)$ (see [9, Theorem 5.9]). Consequently $\{Tf_\gamma\}$ converges uniformly to θ in Y and it follows that T is β_τ-$\|\cdot\|$ continuous.

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