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ON FK-SPACES WHICH ARE BOUNDEDNESS DOMAINS

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ABSTRACT. An open question in summability theory is to characterize those FK-spaces, E, which are boundedness domains (i.e., $E = m_A$ for some infinite matrix A). As a partial solution to this problem we give necessary and sufficient conditions for an FK space E, which has the T-sectional boundedness property, to be equal to m_A for some row-finite A.

1. **Introduction.** An open question in summability theory is to characterize those FK-spaces, E, which are boundedness domains (i.e., $E = m_A$ for some infinite matrix A). In this paper we solve this problem in the case when E has the T-sectional boundedness property (T-AB) and A is row-finite. In addition we give a characterization of those FK-spaces, E, which are closed subspaces of m_A for some row-finite matrix A.

2. **Preliminary ideas and results.** A K-space is a locally convex space of real or complex sequences such that the projection seminorm $p_j(x) = |x_j|$ is continuous for $j = 1, 2, \cdots$. An FK-space is a complete, metrizable K-space. If E is an FK-space and a Banach space we say that E is a BK-space. If the FK topology of E can be generated by the seminorms $\{p_j: j = 1, 2, \cdots\}$ and a single additional seminorm, q, we say that E is an almost BK-space. A continuous seminorm, q, on an almost BK-space E is said to be nontrivial if q and $\{p_j: j = 1, 2, \cdots\}$ generate the topology of E.

Let s be the set of all sequences and let m be the set of all bounded sequences. m is a BK-space with the norm $\|x\|_\infty = \sup_k |x_k|$. Let A be an infinite matrix. Define Ax to be the sequence whose ith term is $\sum_{k=1}^\infty a_{ik} x_k$ provided the series converges. For an FK-space E and an infinite matrix A we define $E_A = \{x \in s: Ax \in E\}$. m_A is called the boundedness domain of A. If A is row-finite, m_A is an almost BK-space with its topology generated by the seminorms $\|Ax\|_\infty$ and $\{p_j: j = 1, 2, \cdots\}$. See [5, §12.4].

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A sequence $x = (x_k)$ is said to be finite if $x_k = 0$ for all but finitely many k. Let E^∞ denote the set of finite sequences. Throughout this paper we will assume that every FK-space, E, contains E^∞. Let $T = (t_{mn})$ be a regular row-finite matrix. Let e^k be the sequence with 1 in the kth position and 0's elsewhere. Define the mth T-section of a sequence x to be the finite sequence

$$T_m x = \sum_{n=1}^{\infty} t_{mn} \sum_{k=1}^{n} x_k e^k.$$

An FK-space E has T-AK if $x = \lim_m T_m x$ with respect to the topology of E; for all $x \in E$. E has T-AB if the set of T-sections of x is bounded for each $x \in E$. Unless otherwise stated, E will denote an FK-space having T-AB. We denote by $E_{T\text{-}AK}$ the set $\{x \in E: x = \lim_m T_m x\}$. If E has T-AB, $E_{T\text{-}AK}$ is a closed subspace of E. The T-AK and T-AB properties are discussed in [3] and [4].

Let A be any coordinatewise bounded subset of s. Define

$$p_{A(T)}(x) = \sup_{m;a \in A} \left| \sum_{k=1}^{\infty} a_k (T_m x)_k \right|.$$

Let $F_{A(T)} = \{x \in s: p_{A(T)}(x) < \infty\}$. By [4, Theorem 4.2] $F_{A(T)}$ is an almost BK-space with its topology genereted by $p_{A(T)}$ and $\{p_j: j = 1, 2, \cdots\}$. If $A = (a_{ik})$ is an infinite matrix, A will be regarded as a countable set of sequences $\{a^i: a^i_k = a_{ik}\}$.

2.1. **Proposition.** *If A is row-finite and m_A has T-AB then $m_A = F_{A(T)}$.*

Proof. If m_A has T-AB, then $p_{A(T)}(x) = \sup \|A(T_m x)\|_\infty < \infty$. Thus $m_A \subseteq F_{A(T)}$. Suppose $x \in F_{A(T)}$. Then

$$\|Ax\|_\infty \le \sup_{i,m} \left| \sum_{n=1}^{\infty} t_{mn} \sum_{k=1}^{n} a_{ik} x_k \right| = p_{A(T)}(x) < \infty.$$

Thus $F_{A(T)} \subseteq m_A$.

For any sequence space E, define

$$E^{\gamma(T)} = \left\{ y \in s: \sup_m \left| \sum_{k=1}^{\infty} y_k (T_m x) \right| < \infty \quad \text{for all } x \in E \right\}.$$

Clearly $E \subseteq E^{\gamma(T)\gamma(T)}$. If $E = E^{\gamma(T)\gamma(T)}$ then E is called a $\gamma(T)$-space. By [4, Proposition 4.3] $F_{A(T)}$ is a $\gamma(T)$-space. As a corollary to the above proposition we have that if m_A has T-AB, then m_A is a $\gamma(T)$-space. Properties of boundedness domains related to $\gamma(T)$-spaces, when T is the identity matrix, have been studied in [1].

Thus, two necessary conditions for an FK-space E with T-AB to be a boundedness domain for some row-finite matrix are: (1) E is an almost BK-space; and (2) E is a $\gamma(T)$-space. We now turn to the task of proving them sufficient. But before we can do this we must further examine the structure of the space $F_{A(T)}$.

3. **The space** $F_{A(T)}$. When E is a T-AK space, it follows from [4, Theorem 3.2] that $E^{\gamma(T)}$ can be identified with the dual space of E. If E has T-AB it then follows from [4, Theorem 3.3] that $E^{\gamma(T)}$ can be identified with the dual space of $E_{T\text{-}AK}$. If E has T-AB, we can then give $E^{\gamma(T)}$ the weak* topology induced by $E_{T\text{-}AK}$. For a set $A \subseteq E^{\gamma(T)}$ we will denote the weak* closed absolutely convex cover of A by $(\Delta A)^-$.

3.1. **Lemma.** *Let E be an FK-space having T-AB. Then* $p_{A(T)}(x) = p_{(\Delta A)^-(T)}(x)$ *for all $x \in s$ and $A \subseteq E^{\gamma(T)}$.*

Proof. Define the bilinear functional

$$u(x, a) = \lim_m \sum_{k=1}^{\infty} a_k (T_m x)_k$$

on $E_{T\text{-}AK} \times E^{\gamma(T)}$. $E_{T\text{-}AK}$ and $E^{\gamma(T)}$ are in duality with respect to this bilinear functional and the weak* topology is compatible with this duality. By [5, §13.2, Theorem 1], $A^{00} = (\Delta A)^-$. It is easily verified that $\sup_{a\in A} |u(x, a)| = \sup_{a \in A^{00}} |u(x, a)|$ for all $x \in E_{T\text{-}AK}$. Now for any $x \in s$, $T_m x \in E_{T\text{-}AK}$. Thus

$$p_{A(T)}(x) = \sup_{m; a\in A} |u(T_m x, a)| = \sup_{m; a \in A^{00}} |u(T_m x, a)| = p_{(\Delta A)^-(T)}(x).$$

As a result, we have that $F_{A(T)} = F_{(\Delta A)^-(T)}$ for any $A \subseteq E^{\gamma(T)}$.

3.2. **Lemma.** E^∞ *is weak* dense in* $E^{\gamma(T)}$.

Proof. Suppose $x \in E_{T\text{-}AK}$ is such that $\lim_m \sum_{k=1}^{\infty} y_k (T_m x)_k = 0$ for all $y \in E^\infty$. Since T is regular, we must have that $\lim_m (T_m x)_i = x_i = 0$ for all $i = 1, 2, \cdots$. Thus $x = 0$. The result then follows from the Hahn-Banach theorem.

4. Main results. Let q be a continuous seminorm on E. Define

$$B_q = \left\{ y \in E^{\gamma(T)}: \sup_{x \in E_{T\text{-}AK};\, q(x) \le 1} \left| \lim_m \sum_{k=1}^{\infty} y_k (T_m x)_k \right| \le 1 \right\}.$$

4.1. Theorem. *Let E be an FK-space which has T-AB.*

(1) *$E = m_A$ for some row-finite matrix A if and only if E is an almost BK-space and E is a $\gamma(T)$-space.*

(2) *E is a closed subspace of m_A for some row-finite matrix A if and only if E is an almost BK-space and there is a number $M > 0$ such that $q(x) \le M \sup_m q(T_m x)$ for all $x \in E$, for some nontrivial seminorm q.*

Proof. The proof of the necessity of (1) was given in §2. Suppose E is an almost BK-space and a $\gamma(T)$-space. Let q be a continuous nontrivial seminorm on E. By the construction in [4, Theorem 4.4] $F_{B_q(T)} = E^{\gamma(T)\gamma(T)} = E$. By Lemma 3.2 there is a countable set of finite sequences, A', such that $(\Delta A')^- = B_q$. Let A be the matrix with the set of rows equal to $\{T_m a: a \in A', m = 1, 2, \cdots\}$. Then $m_A = F_{A'(T)}$ which, by Lemma 3.1, is equal to $F_{B_q(T)}$ which is in turn equal to E.

The necessity of (2) follows since E has T-AB. To show the sufficiency we have from [4, Theorem 4.4] that $E \subseteq E^{\gamma(T)\gamma(T)} = m_A$. By [5, §11.3, Corollary 1] the FK topology of E is stronger than that of m_A. The hypothesis assures that it is weaker. Thus E is closed in m_A.

As a special case we obtain that l^p $(1 \le p \le \infty)$ is a boundedness domain. This result was originally obtained by Bennett in [2, Proposition 9]. We point out that his proof is quite similar to the proof of Theorem 4.1.

4.2. Corollary. *Let E be an almost BK-space having T-AB. Suppose further that E is a $\tau(T)$-space. Let A be a row-finite matrix such that E_A has T-AB. Then E_A is a boundedness domain.*

Proof. By Theorem 4.1, $E = m_B$ for some row-finite matrix B. Since A and B are both row-finite, it is easily verified that $E_A = m_{BA}$.

We remark that the hypothesis that A is row-finite cannot be omitted from the above theorem. Let A be the matrix with 1's in the first row and 0's elsewhere. Then $m_A = cs$, the set of convergent series, which is not a $\gamma(T)$-space.

In [6] an example of a convergence domain, c_A, is given in which c_A does not have T-AB for any regular matrix T. It follows that m_A does not have T-AB for any regular T. Thus our results do not cover all possible boundedness domains for row-finite matrices.

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