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HOLOMORPHIC MAPPINGS OF BOUNDED DISTORTION

A. MARDEN AND S. RICKMAN[1]

ABSTRACT. Nonelementary holomorphic mappings $\mathbf{C}^n \to \mathbf{C}^n$, $n \geq 2$, cause severe geometric distortion near ∞ in sharp contrast to the case $n = 1$ when there is no distortion.

The theory of quasiconformal and more generally quasiregular mappings $\mathbf{R}^m \to \mathbf{R}^m$ has been successful in recent years (see, for example, [4]) in drawing global topological consequences from the condition of uniformly bounded local distortion (the precise definition will be given below). With H. Wu [9] it is natural to ask what this condition of bounded distortion implies for holomorphic mappings $\mathbf{C}^n \to \mathbf{C}^n$, regarding these as special cases of mappings $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$. Wu [7], [8] considered such mappings in the course of his investigation of value distribution theory. The purpose of this note is to point out that the bounded distortion condition, imposed merely in a neighborhood of ∞, in the context of holomorphic mappings of $\mathbf{C}^n$, $n \geq 2$, is extremely restrictive: only affine mappings have this property.

Definition. A continuous map $f\colon D \to \mathbf{R}^m$ of a domain $D \subset \mathbf{R}^m$ is *quasiregular* if (a) f is absolutely continuous on lines with L^m_{loc} generalized partial derivatives, and (b) for some $1 \leq K < \infty$,

$$|f'(x)|^m \leq KJ(x, f), \quad \text{a.e.} \tag{1}$$

Here $f'(x)$ denotes the Jacobian matrix, $|f'(x)|$ its norm as a linear transformation, and $J(x, f)$ its determinant. A quasiregular map which is a homeomorphism is called *quasiconformal*.

Heuristically the definition says that a nonconstant quasiregular map sends tiny spheres to tiny ellipsoids for which the ratio of the longest to shortest axis is uniformly bounded by K.

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For $m \geq 3$ it is true [5] (see also [3]) that a nonconstant quasiregular map $f: D \to \mathbf{R}^m$ for which $K = 1$ is the restriction to D of a Möbius transformation.

Definition. A holomorphic map $F: D \to \mathbf{C}^n$ of a domain $D \subset \mathbf{C}^n$ is called *quasiregular* if the induced map $f: D' \to \mathbf{R}^{2n}$ of the corresponding domain $D' \subset \mathbf{R}^{2n}$ is quasiregular, that is, if f satisfies (1) with $m = 2n$.

We remark that this is the same as Wu's definition of "quasiconformal holomorphic maps" [7, p. 229].

Theorem. *A nonconstant holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$, $n \geq 2$, which is quasiregular in the complement of some polydisk, is of the form $F(x) = Ax + b$ where A is a nonsingular $n \times n$ matrix and $b \in \mathbf{C}^n$.*

Proof. The proof is based on two facts for quasiregular maps f of a domain D in $\mathbf{R}^m$, $m \geq 3$. (a) If it is sufficiently smooth, f is a local homeomorphism [2, p. 95]. (b) If ζ is an isolated boundary point of D and f is a local homeomorphism in D then f can be extended to be a quasiregular local homeomorphism in $D \cup \{\zeta\}$ [1]. Both the definition of quasiregularity and these two facts have natural analogs for the one point compactification of $\mathbf{R}^m$ and it is actually in this context that they will be applied.

In our situation then let $f: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ be the map induced by F. Since f is quasiregular outside a sufficiently large ball we conclude from above that f has a "removable singularity" at ∞. Consequently [6, 18.4] f satisfies a condition

$$|f(x)| \leq C|x|^{\alpha}, \qquad |x| > R, \tag{2}$$

for some $\alpha, C > 0$ and $R < \infty$ ($|\cdot|$ denotes the euclidean norm).

Returning to $F = (F_1, \cdots, F_n)$, (2) implies that each component F_i is a polynomial. The complex Jacobian determinant $\mathcal{J}(x, F)$ is also a polynomial. Therefore there are only two possibilities: either (i) $\{x: \mathcal{J}(x, F) = 0\}$ is an algebraic variety of dimension ≥ 1 and, in particular, is not compact, or (ii) $\mathcal{J}(x, F)$ is a constant $\neq 0$.

Case (i) can be ruled out because F is a local homeomorphism outside a sufficiently large polydisk and hence must have nonvanishing Jacobian there. Consequently case (ii) occurs. Then (1) implies that all the partial derivatives of f and hence of F are constants since they are polynomials (recall that $J(x, f) = |\mathcal{J}(x, F)|^2$). Therefore F has the desired form, completing the proof.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND