PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

HOLOMORPHIC MAPPINGS OF BOUNDED DISTORTION

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ABSTRACT. Nonelementary holomorphic mappings $C^n \to C^n$, $n \geq 2$, cause severe geometric distortion near ∞ in sharp contrast to the case $n = 1$ when there is no distortion.

The theory of quasiconformal and more generally quasiregular mappings $\mathbf{R}^m \to \mathbf{R}^m$ has been successful in recent years (see, for example, [4]) in drawing global topological consequences from the condition of uniformly bounded local distortion (the precise definition will be given below). With H. Wu [9] it is natural to ask what this condition of bounded distortion implies for holomorphic mappings $\mathbf{C}^n \to \mathbf{C}^n$, regarding these as special cases of mappings $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$. Wu [7], [8] considered such mappings in the course of his investigation of value distribution theory. The purpose of this note is to point out that the bounded distortion condition, imposed merely in a neighborhood of ∞, in the context of holomorphic mappings of $\mathbf{C}^n$, $n \geq 2$, is extremely restrictive: only affine mappings have this property.

Definition. A continuous map $f: D \to \mathbf{R}^m$ of a domain $D \subset \mathbf{R}^m$ is *quasiregular* if (a) f is absolutely continuous on lines with L^m_{loc} generalized partial derivatives, and (b) for some $1 \leq K < \infty$,

$$(1) \qquad |f'(x)|^m \leq KJ(x, f), \quad \text{a.e.}$$

Here $f'(x)$ denotes the Jacobian matrix, $|f'(x)|$ its norm as a linear transformation, and $J(x, f)$ its determinant. A quasiregular map which is a homeomorphism is called *quasiconformal*.

Heuristically the definition says that a nonconstant quasiregular map sends tiny spheres to tiny ellipsoids for which the ratio of the longest to shortest axis is uniformly bounded by K.

Received by the editors August 6, 1973.

AMS (MOS) subject classifications (1970). Primary 32H99.

Key words and phrases. Quasiregular holomorphic mapping.

[1]Authors supported in part by the National Science Foundation.

For $m \geq 3$ it is true [5] (see also [3]) that a nonconstant quasiregular map $f: D \to \mathbf{R}^m$ for which $K = 1$ is the restriction to D of a Möbius transformation.

Definition. A holomorphic map $F: D \to \mathbf{C}^n$ of a domain $D \subset \mathbf{C}^n$ is called *quasiregular* if the induced map $f: D' \to \mathbf{R}^{2n}$ of the corresponding domain $D' \subset \mathbf{R}^{2n}$ is quasiregular, that is, if f satisfies (1) with $m = 2n$.

We remark that this is the same as Wu's definition of "quasiconformal holomorphic maps" [7, p. 229].

Theorem. *A nonconstant holomorphic map* $F: \mathbf{C}^n \to \mathbf{C}^n$, $n \geq 2$, *which is quasiregular in the complement of some polydisk, is of the form* $F(x) = Ax + b$ *where* A *is a nonsingular* $n \times n$ *matrix and* $b \in \mathbf{C}^n$.

Proof. The proof is based on two facts for quasiregular maps f of a domain D in $\mathbf{R}^m$, $m \geq 3$. (a) If it is sufficiently smooth, f is a local homeomorphism [2, p. 95]. (b) If ζ is an isolated boundary point of D and f is a local homeomorphism in D then f can be extended to be a quasiregular local homeomorphism in $D \cup \{\zeta\}$ [1]. Both the definition of quasiregularity and these two facts have natural analogs for the one point compactification of $\mathbf{R}^m$ and it is actually in this context that they will be applied.

In our situation then let $f: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ be the map induced by F. Since f is quasiregular outside a sufficiently large ball we conclude from above that f has a "removable singularity" at ∞. Consequently [6, 18.4] f satisfies a condition

$$|f(x)| \leq C|x|^{\alpha}, \qquad |x| > R, \tag{2}$$

for some α, $C > 0$ and $R < \infty$ ($|\cdot|$ denotes the euclidean norm).

Returning to $F = (F_1, \cdots, F_n)$, (2) implies that each component F_i is a polynomial. The complex Jacobian determinant $\mathcal{J}(x, F)$ is also a polynomial. Therefore there are only two possibilities: either (i) $\{x: \mathcal{J}(x, F) = 0\}$ is an algebraic variety of dimension ≥ 1 and, in particular, is not compact, or (ii) $\mathcal{J}(x, F)$ is a constant $\neq 0$.

Case (i) can be ruled out because F is a local homeomorphism outside a sufficiently large polydisk and hence must have nonvanishing Jacobian there. Consequently case (ii) occurs. Then (1) implies that all the partial derivatives of f and hence of F are constants since they are polynomials (recall that $J(x, f) = |\mathcal{J}(x, F)|^2$). Therefore F has the desired form, completing the proof.

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