PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

ON SUBDIRECT PRODUCTS OF RINGS WITHOUT SYMMETRIC DIVISORS OF ZERO

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ABSTRACT. A theorem of V. A. Andrunakievič and Ju. M. Rjabuhin asserts that a ring R is without nilpotent elements if and only if R is a subdirect product of skew-domains. In this paper we prove that a semiprime ring R with involution is a subdirect product of rings without symmetric divisors of zero if and only if R is compressible for its symmetric elements.

Introduction. Let R be a ring with involution $*$, S the set of all symmetric elements ($x = x^*$) of R, T the set of all *traces* ($t + t^*$) of R, N the set of all *norms* (tt^*) of R, and $S_0 = T \cup N \subseteq S$. Given a symmetric ideal $I = I^*$ of R, the factor ring R/I will be equipped with the canonical involution $x + I \longrightarrow x^* + I$. R is said to be *$*$-compressible* if for any $x \in S_0$ and n equal to a power of 2, $ax^n b = 0$ implies $axb = 0$.

Andrunakievič and Rjabuhin have shown that a ring R is without nilpotents if and only if R is a subdirect product of skew-domains [1]. Related to this, one would like to ask when R has a subdirect representation into rings without symmetric divisors of zero. We prove here that a semiprime ring R is $*$-compressible if and only if R is a subdirect product of rings without symmetric divisors of zero (in the case of 2-torsion free).

Remark 1. Let $R = F_2$ be the ring of matrices over a field F. Assume that for a given involution$*$, the symmetric divisors of zero are zero alone. Then $*$ must be the *symplectic* involution (i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$). In fact if $*$ were not the symplectic involution then, by a result of Jacobson [7, Case A, p.311], $*$ must be of the following type:

Presented to the Society, February 19, 1973; received by the editors September 7, 1973.

AMS (MOS) subject classifications (1970). Primary 16A28; Secondary 16A48.

Key words and phrases. Ring with involution, symmetric elements, traces, norms, $*$-prime rings, m-system, $*$-compressible rings.

[1] This research was supported partly by Ontario Graduate Fellowship and partly by Department of Mathematics, Carleton University, Ottawa, Canada.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} r_1^{-1}\bar{a}r_1 & r_1^{-1}\bar{c}r_2 \\ r_2^{-1}\bar{b}r_1 & r_2^{-1}\bar{d}r_2 \end{pmatrix},$$

where F has an involution $\lambda \to \bar{\lambda}$ and $\bar{r}_i = r_i$ $(i = 1, 2)$ are fixed invertible symmetric elements of F. Now take $x = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$; then $x^2 = 0$. Then

$$xx^* = \begin{pmatrix} 0 & 0 \\ 0 & r_1^{-1}r_2 \end{pmatrix} \neq 0 \quad \text{and} \quad x(xx^*) = 0.$$

This means xx^* is a symmetric divisor of zero, a contradiction. This shows that $*$ must be the symplectic one.

Remark 2. C. Lanski has characterized a semiprime ring R such that all its nonzero symmetric elements do not annihilate themselves in the case of 2-torsion free; namely, that R has one of the following types: (i) a skew-domain, (ii) a subring of the direct sum of a skew-domain and its opposite with interchanging co-ordinate involution, or (iii) an order in 2×2 matrices over a field [9]. Montgomery and Herstein extended Lanski's characterization to the case where R is *any* semiprime ring with involution such that all its nonzero traces do not annihilate themselves. In case (i), R is, of course, $*$-compressible. In case (ii), R is a subring of the direct sum of $*$-compressible rings, and, consequently, R is again $*$-compressible. In case (iii), by using Remark 1 we get $S_0 \subseteq Z(F_2)$, the center of F_2, and so R is $*$-compressible. This shows that any semiprime ring with involution such that all its nonzero traces do not annihilate themselves is a $*$-compressible ring.

We first prove several propositions.

Proposition 1. *Let R be a $*$-compressible ring.*

(1) *If $s \in S_0$ with $s^n = 0$, then $s = 0$.*

(2) *If $xx^* = 0$ then $x^*x = 0$.*

Proof. (1) follows immediately from the definition of a $*$-compressible ring. As for (2), note that $(x^*x)^2 = 0$ and apply (1).

Recall that a ring R is *$*$-prime* if for any symmetric ideals $A = A^*$ and $B = B^*$ of R, $AB = 0$ implies $A = 0$ or $B = 0$.

Proposition 2. *Let R be a $*$-prime ring. Then R is $*$-compressible if and only if every nonzero symemtric element in S_0 is a nondivisor of zero.*

Proof. "If" part. This is an immediate consequence of Remark 2. "Only if" part. First we show that $sd = 0$ with $s, d \in S_0$ implies $s = 0$ or $d = 0$.

Let $x \in R$. We have

$$s(sxd + dx^*s)^2 d$$
$$= s(sxdsxd + dx^*s^2xd + sxd^2x^*s + dx^*sdx^*s)d = 0$$
$$\text{(because } ds = sd = 0).$$

By the hypothesis, $s(sxd + dx^*s)d = 0$. This means that $s^2xd^2 = 0$ $\forall x \in R$. Since R is *-prime, $s^2 = 0$ or $d^2 = 0$. Since R is *-compressible, $s = 0$ or $d = 0$. By Lanski's characterization, mentioned in Remark 2, R is without nonzero symmetric divisors of zero in S_0.

Recall that a subset M_0 of a ring R is an *m-system* if $a, b \in M_0$ imply $axb \in M_0$ for some $x \in R$.

Proposition 3. *Let R be a *-compressible ring. Let M_0 be an m-system excluding* (0). *Then* (0) *can be enlarged to a *-prime ideal P such that R/P is *-compressible and $P \cap M_0 = \emptyset$.*

Proof. Let M be an m-system containing M_0 and maximal with respect to the exclusion of the ideal (0). As is well known, if J is the complement of M, $J = C(M)$, then J is a prime ideal of R excluding M_0. Assuming for the moment that J has the following property:

(1) for any $s \in S_0$ and $xs^ny \in J$ which imply $xsy \in J$, where n is a power of 2, we take $P = J \cap J^*$, and so have a *-prime ideal P such that R/P is *-compressible and $P \cap M_0 = \emptyset$.

It remains to show that J has property (1). Define

$$M_1 = M \cup \{xs^ny \mid xsy \in M;\ n = 1, 2, \cdots, 2^h, \cdots\},$$

$$M_2 = M_1 \cup \{x_1s_1^ny_1 \mid x_1s_1y_1 \in M_1;\ n = 1, 2, \cdots, 2^h, \cdots\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M_{k+1} = M_k \cup \{x_ks_k^ny_k \mid x_ks_ky_k \in M_k;\ n = 1, 2, \cdots, 2^h, \cdots\}.$$

Let $M' = \bigcup_k M_k$. Of course, $M' \supseteq M$. Now M' excludes (0), for if M_k excludes (0), M_{k+1} will exclude (0) (since R is *-compressible) and, by construction, M excludes (0). By induction on k, all M_k exclude (0) and consequently M' excludes (0).

Next we show that M' is an m-system. In fact, it is enough to show that for any pair $a, b \in M_k$ there is $c \in R$ such that $acb \in M'$ (for $M \subseteq M_1 \subseteq M_2 \subseteq \cdots$). The property holds for $k = 1$. Now suppose it is true for k; let us show this for $k + 1$. Let $a, b \in M_{k+1}$. We wish to find a $c \in R$ such that $acb \in M'$. There are three cases:

Case 1. Both $a, b \in M_k$. Here, c follows from the induction step.

Case 2. Both $a, b \in \{xs^n y | xsy \in M_k\}$. Here $a = xs^n y$, $b = zd^m t$ for some x, y, t and z such that xsy and $zdt \in M_k$. By the induction hypothesis, there exists $c \in R$ such that $(xsy)c(zdt) \in M_{k'}$ for some k'. Then $xs^n(yczdt) \in M_{k'+1}$ and $(xs^n ycz)d^m t \in M_{k'+2}$. Hence $acb \in M'$.

Case 3. If $a \in M_k$ and $b \in \{xs^n y | xsy \in M_k\}$, then we have $a \in M_k$ and $b = xs^n y$ where $xsy \in M_k$. Thus there exists $c \in R$ such that $ac(xsy) \in M_{k'}$, for some k'. So $(acx)s^n y \in M_{k'+1}$. Hence $acb \in M'$.

This shows that M' is an m-system containing M. By the above, M' excludes (0). By maximality, $M' = M$ and so $J = C(M)$ has property (1). The proof is now complete.

We are now in a position to prove the following key theorem.

Theorem 1. *Let R be *-compressible, and η the prime radical of R. Then $\eta = \bigcap\{P' | P'$ is a *-prime ideal of R and R/P' has no nonzero symmetric divisors of zero in $S_0\}$.*

Proof. If P is a prime ideal of R, then $M = C(P)$ is an m-system excluding (0). By Proposition 3, the ideal (0) can be enlarged to a *-prime ideal P' such that R/P' is *-compressible and $P' \cap M = \emptyset$. Thus for such a *-prime ideal P', we have $P' \subseteq C(M) = P$ and R/P' has no nonzero symmetric divisors of zero in S_0 (Proposition 2). Hence $\eta \supseteq \bigcap\{P' | P'$ is a *-prime ideal of R and R/P' has no nonzero symmetric divisors of zero in $S_0\}$. Since η is the least semiprime ideal of R, we have the equality; hence the proof is complete.

We can now derive our main theorem.

Theorem 2. *Let R be a semiprime ring with involution. Then R is *-compressible if and only if R is a subdirect product of rings without nonzero symmetric divisors of zero in S_0.*

Proof. "Only if" part. It follows immediately from Theorem 1. "If" part. Let R be a subdirect product of rings without symmetric divisors of zero in S_0. Let $x, y \in R$ and $s \in S_0$ with $xs^n y = 0$. We prove that $xsy = 0$. In fact, in each factor R_i, we have $x_i s_i^n y_i = 0$. Since R_i is semiprime and without symmetric divisors of zero in S_0, R_i is *-compressible (Remark 2). Thus $x_i s_i y_i = 0$ for all indices i. Therefore $xsy = 0$.

Corollary 1. *Let R be a semiprime and *-compressible ring. Then R is a subring of a direct product of rings which are either skew-domains or orders in 2×2 matrices over a field.*

Corollary 2. *A semiprime ring R is *-compressible if and only if $xs^n y = 0$ implies $xsy = 0$ for any $s \in S_0$ and any integer $n > 1$.*

Remark 3. If R is 2-torsion free, then we observe that if S_0 does not contain nonzero divisors of zero, then S itself does not contain nonzero divisors of zero. Also, if M is an m-system excluding (0) then $M' = \{2^k x | x \in M, k = 1, 2, \cdots\}$ is also an m-system excluding (0). For any *-prime ideal P with $P \cap M' = \emptyset$, R/P is 2-torsion free (for if $2x \in P$ with $x \notin P$ then $2R(x, x^*) \subseteq P$; since P is *-prime we have $2R \subseteq P$, a contradiction).

Combining Remark 3 with Proposition 2 and Theorem 1, we have

Theorem 3. *Let R be a 2-torsion free and semiprime ring. Then R is *-compressible if and only if R is a subdirect product of rings without nonzero symmetric divisors of zero.*

From Theorem 2 we derive the following theorems which generalize [8], [12].

Theorem 4. *Let R be a semiprime and *-compressible ring. If $s_1 s_2 \cdots s_n = 0$ with $s_i \in S_0$, then for any permutation $i_1, i_2, \cdots, i_n$ in the i's $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$ (that is, the product of the s_i's is zero in any order).*

Proof. Let $({}_\lambda s_i)_{\lambda \in I}$ be the image of s_i under a subdirect representation as in Theorem 2. If $s_1 s_2 \cdots s_n = 0$, then ${}_\lambda s_1 \, {}_\lambda s_2 \cdots {}_\lambda s_n = 0$ for all $\lambda \in I$. Using the regularity condition in each factor ${}_\lambda R$ of R, there must be ${}_\lambda s_i = 0$. Thus ${}_\lambda s_{i_1} \, {}_\lambda s_{i_2} \cdots {}_\lambda s_{i_n} = 0$ for all $\lambda \in I$. Consequently $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$.

Remark 4. Let R be as in Theorem 4. By using Theorem 2 and Lanksi's theorem, one can see that if $sxdyt = 0$ with $s, d, t \in S_0$ and $x, y \in R$, then $sdtxy = 0$.

R is said to be **-von Neumann regular* (*-regular) if for any $s \in S_0$ there is $x \in R$ such that $a = axa$.

Theorem 5. *Let R be as in Theorem 4. Assume that R/P is *-regular for every *-prime ideal P. Then R is *-regular.*

Proof. Let $a \in S_0$ and E be the set of all elements of the form

$$(a - ax_1 a)(a - ax_2 a) \cdots (a - ax_n a);$$

$x_1, x_2, \cdots, x_n$ running over R. Clearly E is closed under multiplication. Then E is an m-system. We claim that $0 \in E$. For if $0 \notin E$, then by Prop-

osition 3, the ideal (0) can be enlarged to a *-prime ideal P such that R/P is *-compressible and $P \cap E = \emptyset$. By the hypothesis, R/P is a *-regular ring and consequently $a - aya \in P$ for a suitable y. But $a - aya \in E$, a contradiction. We must conclude that $0 \in E$, that is, for some $x_i \in R$,

$$a(1 - x_1 a)a(1 - x_2 a) \cdots a(1 - x_n a) = 0.$$

By Remark 4, we have

$$a^n(1 - x_1 a) \cdots (1 - x_n a) = 0.$$

Thus $a(1 - x_1 a) \cdots (1 - x_n a) = 0$. Since the product $(1 - x_1 a) \cdots (1 - x_n a)$ has the form $1 - za$, $a(1 - za) = 0$, that is, $a = aza$.

Remark 5. A long standing conjecture of I. Kaplansky [8] was that a ring R is von Neumann regular iff R is a semiprime ring such that each prime image of R is von Neumann regular. This conjecture was settled in the affirmative by I. N. Herstein and E. T. Wong separately in the case of rings without nilpotent elements [12] (see also Fisher and Snider [3]). Theorem 5, whose proof follows the pattern of Herstein-Kaplansky's proof (see [8]), is a generalization of Wong's result.

Acknowledgements. This paper is a part of the author's doctoral dissertation at Carleton University. The author gratefully acknowledges Professor Maurice Chacron for his advice and assistance.

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