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SYLOWIZERS IN LOCALLY FINITE GROUPS

M. J. TOMKINSON

ABSTRACT. A result of W. Gaschütz for finite soluble groups is extended to two classes of locally finite, locally soluble groups.

The concept of Sylowizers has been introduced by W. Gaschütz [2]. If R is a π-subgroup of the group G, then a *π-Sylowizer* of R in G is a subgroup S of G maximal with respect to containing R as a Sylow π-subgroup. [π denotes a set of primes and a Sylow π-subgroup is simply a maximal π-subgroup.] A straightforward Zorn's lemma argument shows that π-Sylowizers of any π-subgroup R must always exist. Gaschütz proved the following conjugacy theorem:

Let G be a finite soluble group and R a normal subgroup of some Sylow π-subgroup P of G. Then the π-Sylowizers of R in G are conjugate in G.

It is our aim in this note to extend this result to the class $\mathfrak{G}$ of periodic locally soluble FC-groups and the class $\mathfrak{U}$ defined by:

$G \in \mathfrak{U}$ if and only if G is locally finite and for each $H \leq G$ and for each set of primes π, the Sylow π-subgroups of H are conjugate in H. [The necessary results about $\mathfrak{G}$ and $\mathfrak{U}$ may be found in [5] and [1] respectively.]

The proof for finite groups involves the usual consideration of a counterexample of minimal order. This method cannot be employed for infinite groups and we make a more direct construction of the π-Sylowizers of R, although this construction is based on the ideas used in Gaschütz's proof.

Although we only prove the extension of the theorem for $\mathfrak{G}$ and $\mathfrak{U}$, the construction of the π-Sylowizers is carried out in a much wider class of groups. We define $\mathfrak{X}_\pi$ to be the class of upper π-separable locally finite groups G such that PK/K is a Sylow π-subgroup of H/K whenever $K \triangleleft H \leq G$ and P is a Sylow π-subgroup of H.

A $\mathfrak{G}$-group is clearly upper π-separable and also its Sylow π-subgroups have the necessary homomorphism property [5, 4.1 (iii)]. Also $\mathfrak{U}$-groups are

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upper π-separable for all π [3, Theorem E] and the second property is given in [1, 2.1 (ii)]. Thus $\mathfrak{X}_\pi$ is a QS-closed class containing both $\mathfrak{G}$ and $\mathfrak{U}$.

In the construction of the Sylowizers of R we shall require the following elementary result stated in [2] for finite groups.

Lemma 1. *Let R be a π-subgroup of the $\mathfrak{X}_\pi$-group G and $N \lhd G$.*

(a) *If $N \leq R$ then S is a π-Sylowizer of R in G if and only if $N \leq S$ and S/N is a π-Sylowizer of R/N in G/N.*

(b) *If N is a π'-group then S is a π-Sylowizer of R in G if and only if $N \leq S$ and S/N is a π-Sylowizer of RN/N in G/N.*

This lemma shows immediately in the finite case that if G is a minimal counterexample to the theorem then $O_{\pi'}(G) = 1$ and there are no normal subgroups of G contained in R. We show that in any $\mathfrak{X}_\pi$-group we can consider a section G_ρ/R_ρ and the π-subgroup RR_ρ/R_ρ with these properties.

We define four chains of subgroups in G depending only on the π-subgroup R. These chains are defined inductively as follows:

$R_0 = 1$, $G_0 = G$, $Q_0 = O_{\pi'}(G)$ and $P_0 = O_{\pi'\pi}(G)$.

$R_{\alpha+1} = P_\alpha \cap RQ_\alpha$, $G_{\alpha+1} = N_{G_\alpha}(R_{\alpha+1})$, $Q_{\alpha+1}/R_{\alpha+1} = O_{\pi'}(G_{\alpha+1}/R_{\alpha+1})$ and $P_{\alpha+1}/R_{\alpha+1} = O_{\pi'\pi}(G_{\alpha+1}/R_{\alpha+1})$.

Clearly

$$R_{\alpha+1} \geq R_\alpha, \quad G_{\alpha+1} \leq G_\alpha, \quad Q_{\alpha+1} \geq R_{\alpha+1} \geq Q_\alpha \text{ and } P_{\alpha+1} \geq P_\alpha.$$

If α is a limit ordinal, we define

$$R_\alpha = \bigcup_{\beta<\alpha} R_\beta, \quad G_\alpha = \bigcap_{\beta<\alpha} G_\beta, \quad Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta \text{ and } P_\alpha = \bigcup_{\beta<\alpha} P_\beta.$$

It is clear that, for each ordinal α, R_α, Q_α and P_α are normal subgroups of G_α and $R_\alpha \leq Q_\alpha \leq R_{\alpha+1} \leq P_\alpha$.

Let ρ be the first ordinal such that $R_\rho = R_{\rho+1}$. (Such an ordinal must exist, for otherwise $|R_\alpha|$ would be greater than or equal to the cardinality of α, for all α, and this is clearly not possible if the cardinality of α is greater than $|G|$.) Then $G_{\rho+1} = N_{G_\rho}(R_{\rho+1}) = G_\rho$, $Q_{\rho+1}/R_\rho = O_{\pi'}(G_\rho/R_\rho)$ and so $Q_{\rho+1} = Q_\rho$ and similarly $P_{\rho+1} = P_\rho$. Thus all four chains become stationary at the ordinal $\rho = \rho(R)$.

Lemma 2. *G_ρ contains $N_G(R)$ and R is a Sylow π-subgroup of RR_ρ.*

Proof. By induction on ρ. The result is clearly true if $\rho = 0$ and so we may assume that $N_G(R) \leq G_\alpha$ and R is a Sylow π-subgroup of RR_α for each $\alpha < \rho$.

If ρ is a limit ordinal, then the result follows easily from $G_\rho = \bigcap_{\alpha<\rho} G_\alpha$ and $R_\rho = \bigcup_{\alpha<\rho} R_\alpha$. Therefore we may assume that $\rho = \alpha + 1$.

Since $N_G(R) \leq G_\alpha$, $N_G(R)$ normalizes P_α and Q_α and hence normalizes $R_\rho = P_\alpha \cap RQ_\alpha$. Thus $N_G(R) \leq G_\rho$.

$RR_\rho = R(P_\alpha \cap RQ_\alpha) = R(P_\alpha \cap R)Q_\alpha = RQ_\alpha$. Q_α/R_α is a π'-group and R is a Sylow π-subgroup of RR_α. It follows easily that R is a Sylow π-subgroup of $RQ_\alpha = RR_\rho$.

Lemma 3. *Let R be a π-subgroup of the $\mathfrak{X}_\pi$-group G and S a π-Sylowizer of R in G. If $K \leq S \leq H$ and $K \lhd H$, then S/K is a π-Sylowizer of RK/K in H/K.*

Proof. Since R is a Sylow π-subgroup of S, it follows that RK/K is a Sylow π-subgroup of S/K. To show that S/K is maximal in H/K with this property, suppose that RK/K is a Sylow π-subgroup of T/K and $S \leq T \leq H$. Let R^* be a Sylow π-subgroup of T containing R. Then R^*K/K is a π-subgroup of T/K containing RK/K. Thus $R^*K = RK$ and $R \leq R^* \leq RK \leq S$. But R is a Sylow π-subgroup of S and so $R = R^*$ is a Sylow π-subgroup of T. By the maximality of S, we have $T = S$, as required.

We are now able to show that we need only consider the section G_ρ/R_ρ.

Theorem 1. *Let R be a π-subgroup of the $\mathfrak{X}_\pi$-group G. Then, with the notation above, S is a π-Sylowizer of R in G if and only if $R_\rho \leq S \leq G_\rho$ and S/R_ρ is a π-Sylowizer of RR_ρ/R_ρ in G_ρ/R_ρ.*

Proof. Suppose first that S is a π-Sylowizer of R in G. By Lemma 3, it is sufficient to show that $R_\rho \leq S \leq G_\rho$. We prove this by induction on ρ. It is clearly true if $\rho = 0$ and so we may assume that $R_\alpha \leq S \leq G_\alpha$, for all ordinals $\alpha < \rho$. If ρ is a limit ordinal, then the result follows easily from $G_\rho = \bigcap_{\alpha<\rho} G_\alpha$ and $R_\rho = \bigcup_{\alpha<\rho} R_\alpha$. Therefore we may assume that $\rho = \alpha + 1$.

$R_\alpha \leq S \leq G_\alpha$ and so, by Lemma 3, S/R_α is a π-Sylowizer of RR_α/R_α in G_α/R_α. Q_α/R_α is a normal π'-subgroup of G_α/R_α and so $Q_\alpha \leq S$ (Lemma 1 (ii)). Thus $S \geq RQ_\alpha \geq R_{\alpha+1} = R_\rho$.

RQ_α/Q_α is a Sylow π-subgroup of S/Q_α and so the normal π-subgroup $(S \cap P_\alpha)/Q_\alpha$ of S/Q_α is contained in RQ_α/Q_α. Hence $S \cap P_\alpha = RQ_\alpha \cap P_\alpha = R_\rho$ and so $R_\rho \lhd S$, i.e. $S \leq G_\rho$.

Conversely, let S/R_ρ be a π-Sylowizer of RR_ρ/R_ρ in G_ρ/R_ρ. Since R is a Sylow π-subgroup of RR_ρ (Lemma 2), it follows that R is a Sylow π-subgroup of S. Thus S is contained in a π-Sylowizer T of R in G. By the first part of the proof, $T \leq G_\rho$ and T/R_ρ is a π-Sylowizer of RR_ρ/R_ρ in G_ρ/R_ρ. Hence $T = S$, as required.

Theorem 2. *Let R be a normal subgroup of the Sylow π-subgroup P of* *the $\mathfrak{X}_\pi$-group G. Then S is a π-Sylowizer of R in G if and only if S/R_ρ is a Sylow π′-subgroup of G_ρ/R_ρ.*

Proof. Since $Q_\rho = R_\rho = Q_{\rho+1}$, G_ρ/R_ρ has no normal π′-subgroups and so $P_\rho/R_\rho = O_\pi(G_\rho/R_\rho)$. Since $N_G(R) \leq G_\rho$ (Lemma 2), we have $P \leq G_\rho$. PR_ρ/R_ρ is a Sylow π-subgroup of G_ρ/R_ρ and so contains the normal π-subgroup P_ρ/R_ρ. P_ρ and RR_ρ are both normal in PR_ρ and so $[P_\rho, RR_\rho] \leq P_\rho \cap RR_\rho = R_\rho$, i.e. RR_ρ centralizes P_ρ/R_ρ. But $C_{G_\rho}(P_\rho/R_\rho) \leq P_\rho$ [3, Lemma 3.1] and so $RR_\rho \leq P_\rho$ and $R_\rho = R_{\rho+1} = RR_\rho \geq R$. Therefore S/R_ρ is a π-Sylowizer of the trivial subgroup in G_ρ/R_ρ and so is a Sylow π′-subgroup of G_ρ/R_ρ.

Since the Sylow π′-subgroups of a $\mathfrak{G}$-group are locally conjugate and the Sylow π′-subgroups of a $\mathfrak{U}$-group are conjugate we immediately have the following corollaries.

Corollary 1. *If R is a normal subgroup of a Sylow π-subgroup of a $\mathfrak{G}$-group G, then the π-Sylowizers of R in G are locally conjugate in G.*

Corollary 2. *If R is a normal subgroup of a Sylow π-subgroup of a $\mathfrak{U}$-group G, then the π-Sylowizers of R in G are conjugate in G.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GLASGOW, G12 8QW SCOTLAND