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NOTE ON A FAMILY OF VOLTERRA EQUATIONS

KENNETH B. HANNSGEN[1]

ABSTRACT. We prove that the solutions of a certain family of Volterra integrodifferential equations are uniformly bounded. We use this result to determine the asymptotic behavior of the solution of a Volterra equation in Hilbert space.

1. Statement of results. Let $u(t, \lambda)$ denote the solution of

$$u'(t, \lambda) + \lambda \int_0^t a(t-s)u(s, \lambda)\,ds = 0, \qquad u(0, \lambda) = 1 \tag{1.1}$$

(primes denote differentiation with respect to the first variable).

Recently D. F. Shea and S. Wainger [9] proved that

$$\int_0^\infty |u(t, \lambda)|\,dt < \infty \tag{1.2}$$

if $\lambda > 0$ and if $a(t)$ satisfies the conditions

$$\begin{gathered} a(t) \text{ is nonnegative, nonincreasing, and convex on } (0, \infty), \\ \int_0^1 a(t)\,dt < \infty, \text{ and } a(t) \not\equiv a(\infty), \end{gathered} \tag{1.3}$$

$$\lambda a^*(\zeta) \equiv \int_0^\infty e^{-\zeta t}\lambda a(t)\,dt \neq -\zeta \qquad (\mathrm{Re}\ \zeta \geq 0). \tag{1.4}$$

Using a simple identity based on the resolvent formula for (1.1), we prove the following estimate.

Theorem 1. *If* (1.3) *holds and* $\lambda_0 > 0$, *then*

$$|w(t, \lambda)| \leq 4\int_0^\infty |u(t, \lambda_0)|\,dt \qquad (\lambda_0 \leq \lambda < \infty,\ 0 \leq t < \infty), \tag{1.5}$$

where $w(t, \lambda) = \int_0^t u(s, \lambda)\,ds$.

This extends the results in [5], where we required (among other things) that $a(t)$ be twice differentiable if $a \notin L^1(0, \infty)$. The extension is significant because it takes in the special case where

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$$a(t) \text{ is piecewise linear with changes of slope only at integral multiples of } t = t_0 > 0. \tag{1.6}$$

As we showed in [4],

$$u(t, \lambda) \to 0 \quad \text{and} \quad \lambda w(t, \lambda) \to A \equiv \left(\int_0^\infty a(t)\,dt\right)^{-1} \geq 0 \qquad (t \to \infty) \tag{1.7}$$

if (1.3) holds, unless (1.6) holds and

$$\lambda = \lambda_j \equiv j^2[4\pi^2/a(0)t_0^2] \qquad (j = 1, 2, \cdots). \tag{1.8}$$

In the latter case, we set

$$\gamma = 3 - [a(\infty)/a(0)], \qquad \omega_j = (\lambda_j a(0))^{1/2} = 2\pi j/t_0,$$

$$u_j(t) = 2\gamma^{-1} \cos \omega_j t, \qquad w_j(t) = 2[\gamma\omega_j]^{-1} \sin \omega_j t + A/\lambda.$$

Then (1.6) and (1.8) imply

$$u(t, \lambda_j) - u_j(t) \to 0, \qquad w(t, \lambda_j) - w_j(t) \to 0 \qquad (t \to \infty). \tag{1.9}$$

For reference we note that when (1.3) holds

$$|u(t, \lambda)| \leq \sqrt{2} \qquad (0 < \lambda < \infty,\ 0 \leq t < \infty), \tag{1.10}$$

but that $\sup_{0<\lambda, 0\leq t} |w(t, \lambda)| = \infty$. See [5] for the proofs of these facts. We also remark that (1.4) holds whenever (1.3) holds, except when (1.6) and (1.8) hold [4, Lemma 5]. On the other hand, Shea and Wainger proved (1.2) under conditions weaker than those stated above.

We may use Theorem 1 to study the Volterra equation

$$y(t) + \mathbf{L} \int_0^t h(t-s) y(s)\,ds = \xi + t\kappa, \tag{1.11}$$

where $\mathbf{L}$ is a selfadjoint linear operator defined on the dense subspace $\mathfrak{D}$ of the Hilbert space $\mathcal{H}$, $\mathbf{L} \geq \lambda_0 \mathbf{I}$ ($\lambda_0 > 0$), and $h(t) = \int_0^t a(s)\,ds$. Here ξ and κ are prescribed elements of $\mathcal{H}$. As in [6], we may use (1.5) and (1.10) to deduce the representation

$$y(t) = \int_{\lambda_0}^\infty d\mathbf{E}_\lambda [u(t, \lambda)\xi + w(t, \lambda)\kappa] \tag{1.12}$$

($\{\mathbf{E}_\lambda\}$ is the spectral family corresponding to $\mathbf{L}$, $\mathbf{E}_\lambda = \mathbf{E}_{\lambda-}$) and determine the asymptotic behavior of $y(t)$.

Remark. For fixed t, $u(t, \lambda)$ and $w(t, \lambda)$ are continuous functions of λ, even at $\lambda = \lambda_j$ when (1.6) holds. This is shown by an elementary argument (similar to that for a linear ordinary differential equation with a regular linear parameter) using (1.1) and the easily checked equation

$$w'(t, \lambda) + \lambda \int_0^t a(t-s) w(s, \lambda)\,ds = 1. \tag{1.13}$$

This justifies our use of integrals like (1.12).

Theorem 2. *Suppose* (1.3) *holds and let*

$$\Omega(t) = A\mathbf{L}^{-1}\kappa + \sum_{j=1}^{\infty} \mathbf{F}_j[u_j(t)\xi + w_j(t)\kappa],$$

where $\mathbf{F}_j = \mathbf{E}_{\lambda+} - \mathbf{E}_{\lambda}$, $\lambda = \lambda_j$, *and the sum is omitted unless* (1.6) *holds. Then*

$$\lim_{t\to\infty} \|y(t) - \Omega(t)\| = 0. \tag{1.14}$$

Theorem 2 was proved in [6] with smoothness restrictions excluding (1.6) and hence without the sum in $\Omega(t)$. Other recent studies of equations like (1.11) with hypotheses similar to ours include those of C. M. Dafermos [1] and of R. C. MacCamy and J. S. W. Wong [8]. Dafermos discusses applications to viscoelasticity. A. Friedman and M. Shinbrot [3] and later Friedman [2] used spectral decomposition methods to study more general classes of equations in Banach spaces.

Recently [7] we proved a restricted analogue of Theorem 1 for (1.1) with $\lambda a(t)$ replaced by $a(t, \lambda)$; this result also had implications for certain equations in Hilbert space.

2. **Proof of Theorem 1.** Let $\lambda \geq \lambda_0$. Using Laplace transforms or otherwise, one verifies that

$$u(t, \lambda) = u(t, \lambda_0) + \frac{\lambda - \lambda_0}{\lambda} \int_0^t u'(t - s, \lambda)u(s, \lambda_0)\, ds.$$

Therefore,

$$w(t, \lambda) = w(t, \lambda_0) + \frac{\lambda - \lambda_0}{\lambda} \int_0^t [u(t - s, \lambda) - 1]u(s, \lambda_0)\, ds. \tag{2.1}$$

Since $0 < (\lambda - \lambda_0)/\lambda < 1$, (1.10) and (2.1) yield (1.5).

3. **Proof of Theorem 2.** If (1.6) does not hold, the proof is the same as that given in [6] so we assume here that (1.6) holds. To shorten the formulas we also assume $A = 0$. Without loss of generality we take $\lambda_0 < \lambda_1$. Then (1.2) holds with $\lambda = \lambda_0$.

Using (1.1), (1.13), (1.5), and (1.10), one sees by direct substitution that (1.12) gives the unique continuous solution of (1.11). (See [6] for details of this argument.)

We shall also need the following continuity result from [6]. (See "Remarks on piecewise linear kernels.")

Theorem A. *If* (1.3) *and* (1.6) *hold,* $u(t, \lambda)$ *and* $w(t, \lambda)$ *are continuous in* λ, *uniformly in* $0 \leq t < \infty$, *except at* $\lambda = \lambda_j$ $(j = 1, 2, 3, \cdots)$, *where the continuity is not uniform.*

By (1.5) and (1.10), $B = \sup_{t \geq 0, \lambda \geq \lambda_0}(|u(t, \lambda)| + |w(t, \lambda)|) < \infty$. Let $\epsilon > 0$, and choose a positive integer J and a positive number δ such that

$$\int_\Delta [d(\mathbf{E}_\lambda \xi, \xi) + d(\mathbf{E}_\lambda \kappa, \kappa)] < \epsilon, \tag{3.1}$$

where

$$\Delta = \bigcup_{j=1}^{J} ([\lambda_j - \delta, \lambda_j] \cup [\lambda_j +, \lambda_j + \delta]) \cup [\lambda_J + \delta, \infty), \tag{3.2}$$

and δ is taken small enough to make $\lambda_j - \delta > \lambda_{j-1} + \delta$ $(j = 1, 2, \cdots)$. Next set $\Delta_1 = [\lambda_0, \lambda_1 - \delta]$, $\Delta_j = [\lambda_{j-1} + \delta, \lambda_j - \delta]$ $(j = 2, 3, \cdots, J)$. Then

$$\begin{aligned} y(t) - \Omega(t) = & \int_\Delta d\mathbf{E}_\lambda[u(t, \lambda)\xi + w(t, \lambda)\kappa] \\ & + \sum_{j=1}^{J} \mathbf{F}_j\{[u(t, \lambda_j) - u_j(t)]\xi + [w(t, \lambda_j) - w_j(t)]\kappa\} \\ & + \sum_{j=1}^{J} \int_{\Delta_j} d\mathbf{E}_\lambda[u(t, \lambda)\xi + w(t, \lambda)\kappa] - \sum_{j=J+1}^{\infty} \mathbf{F}_j[u_j(t)\xi + w_j(t)\kappa]. \end{aligned} \tag{3.3}$$

By (1.7), (1.9), and Theorem A, there is a $T > 0$ such that $t \geq T$ implies

$$|u(t, \lambda_j) - u_j(t)| + |w(t, \lambda_j) - w_j(t)| < \epsilon \qquad (j = 1, 2, \cdots, J),$$

and $|u(t, \lambda)| + |w(t, \lambda)| < \epsilon$ $(\lambda \in \Delta_1 \cup \cdots \cup \Delta_J)$.

Then the two finite sums in (3.3) are bounded in norm by $2\epsilon(\|\xi\| + \|\kappa\|)$ $(t \geq T)$. By (3.1) the other two terms in (3.3) are bounded in norm by $2\epsilon[B + \gamma^{-1}(1 + \omega_1^{-1})]$. Since ϵ was arbitrary, this establishes (1.14).

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DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061