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A ZERO SET FOR $A(U^2)$

CARL STEPHEN DAVIS

ABSTRACT. A zero set for $A(U^2)$ is constructed using a strictly increasing function on $[0, 1]$ with a derivative that is zero almost everywhere.

1. We let C denote the complex numbers, $U = \{z \in C: |z| < 1\}$, $T = \{w \in C: |w| = 1\}$, $U^2 = U \times U$, and $T^2 = T \times T$. $A(U^2)$ consists of all f analytic on U^2 and continuous on the closure, $\overline{U}^2$ (a similar definition for $A(U)$). A compact subset $K \subset T^2$ is a zero set for $A(U^2)$ if there exists $f \in A(U^2)$ such that $K = \{z \in \overline{U}^2: f(z) = 0\}$. A compact subset $K \subset T^2$ is an interpolation set for $A(U^2)$ if for any $g \in C(K)$ (continuous functions on K) there exists $f \in A(U^2)$ such that $f = g$ on K. A compact subset $K \subset T^2$ is a null set for $A(U^2)$ if for any complex Borel measure μ on T^2 such that $\mu \perp A(U^2)$ (i.e., $\int f\,d\mu = 0$ for all $f \in A(U^2)$), then $|\mu|(K) = 0$. We have similar definitions for $A(U)$ and compact subsets of T.

It is shown in [1] that the three properties above for compact subsets of T^2 (or T) are equivalent. A consequence of the F. and M. Riesz theorem is that $K \subset T$ is a null set for $A(U)$ if and only if $m(K) = 0$, where m is normalized Lebesgue measure on T.

We define $\phi_1: [0, 1] \to T$ by $\phi_1(x) = \exp ix$ and $\phi_2: [0, 1] \times [0, 1] \to T^2$ by $\phi_2((x_1, x_2)) = (\phi_1(x_1), \phi_1(x_2))$. In the following theorem we assume that $J \subset [0, 1]$, $m(J) > 0$, f is analytic in an open neighborhood of J in C, f is real valued and strictly increasing on J, and $f(J) \subset [0, 1]$. A consequence of 6.3.4 in [1] is the following.

Theorem 1. *With f and J as above, let $\psi(x) = (x, f(x))$. Then $\phi_2(\psi(J))$ is not a zero set for $A(U^2)$.*

In [1] it is mentioned that the necessity of the analytic condition on f is an open problem. In this note it is shown that "analytic" cannot be replaced by "continuous". W. Rudin has informed me that L. Carleson (unpublished) has shown that if f is 3 times continuously differentiable, then the conclusion of the theorem holds for J an interval.

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2. For $i = 1, 2$ we define $\pi_i: [0, 1] \times [0, 1] \to T$ by $\pi_i((x_1, x_2)) = \phi_1(x_i)$.

Theorem 2. *Suppose that* $f: [0, 1] \to [0, 1]$ *is strictly increasing and such that* $X = \{(x, f(x)): x \in [0, 1]\} = A_1 \cup A_2$ *where* $m(\pi_i(A_i)) = 0$ *for* $i = 1$ *and* 2. *Then* $|\mu|(\phi_2(X)) = 0$ *for any* $\mu \perp A(U^2)$.

Proof. Let $K \subset \phi_2(A_1)$ be compact. Define $\tilde{K} = \phi_2^{-1}(K) \cap A_1$. Since $m(\pi_1(\tilde{K})) = 0$, $\pi_1(\tilde{K})$ is an interpolation set for $A(U)$. Suppose that $g \in C(K)$. Since G defined by $G(e^{ix_1}) = g((e^{ix_1}, e^{ix_2}))$, where (e^{ix_1}, e^{ix_2}) is the unique element of K having e^{ix_1} as its first component, is in $C(\pi_1(\tilde{K}))$ and $\pi_1(\tilde{K})$ is an interpolation set for $A(U)$, there is an $H \in A(U)$ such that $H = G$ on $\pi_1(\tilde{K})$. Define $h \in A(U^2)$ by $h(z_1, z_2) = H(z_1)$. Then $h = g$ on K. Thus K is an interpolation set for $A(U^2)$. Thus if $\mu \perp A(U^2)$, $|\mu|(K) = 0$. Hence $|\tilde{\mu}|(\phi_2(A_1)) = 0$ (where $\tilde{\mu}$ is the completion of μ). Similarly $|\tilde{\mu}|(\phi_2(A_2)) = 0$, and hence $|\mu|(\phi_2(X)) = 0$.

To show that continuity is not sufficient in Theorem 1, we apply Theorem 2 to a strictly increasing continuous f such that $f'(x) = 0$ for almost all x (dm). We choose $A_2 = \{(x, f(x)): f'(x) = 0\}$ and A_1 the remainder of the graph of f. The equivalence between null sets and zero sets yields the desired result.

REFERENCE

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HAWAII 96822

Current address: Durham Academy Upper School, Durham, North Carolina 27705