A ZERO SET FOR $A(U^2)$

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ABSTRACT. A zero set for $A(U^2)$ is constructed using a strictly increasing function on $[0, 1]$ with a derivative that is zero almost everywhere.

1. We let $C$ denote the complex numbers, $U = \{z \in C: |z| < 1\}$, $T = \{w \in C: |w| = 1\}$, $U^2 = U \times U$, and $T^2 = T \times T$. $A(U^2)$ consists of all $f$ analytic on $U^2$ and continuous on the closure, $\overline{U^2}$ (a similar definition for $A(U)$).

A compact subset $K \subset T^2$ is a zero set for $A(U^2)$ if there exists $f \in A(U^2)$ such that $K = \{z \in \overline{U^2}: f(z) = 0\}$. A compact subset $K \subset T^2$ is an interpolation set for $A(U^2)$ if for any $g \in C(K)$ (continuous functions on $K$) there exists $f \in A(U^2)$ such that $f = g$ on $K$. A compact subset $K \subset T^2$ is a null set for $A(U^2)$ if for any complex Borel measure $\mu$ on $T^2$ such that $\mu 1 A(U^2)$ (i.e., $\int f d\mu = 0$ for all $f \in A(U^2)$), then $|\mu|(K) = 0$. We have similar definitions for $A(U)$ and compact subsets of $T$.

It is shown in [1] that the three properties above for compact subsets of $T^2$ (or $T$) are equivalent. A consequence of the F. and M. Riesz theorem is that $K \subset T$ is a null set for $A(U)$ if and only if $m(K) = 0$, where $m$ is normalized Lebesgue measure on $T$.

We define $\phi_1: [0, 1] \rightarrow T$ by $\phi_1(x) = \exp ix$ and $\phi_2: [0, 1] \times [0, 1] \rightarrow T^2$ by $\phi_2((x_1, x_2)) = (\phi_1(x_1), \phi_1(x_2))$. In the following theorem we assume that $J \subset [0, 1]$, $m(J) > 0$, $f$ is analytic in an open neighborhood of $J$ in $C$, $f$ is real valued and strictly increasing on $J$, and $f(J) \subset [0, 1]$. A consequence of 6.3.4 in [1] is the following.

Theorem 1. With $f$ and $J$ as above, let $\psi(x) = (x, f(x))$. Then $\phi_2(\psi(J))$ is not a zero set for $A(U^2)$.

In [1] it is mentioned that the necessity of the analytic condition on $f$ is an open problem. In this note it is shown that "analytic" cannot be replaced by "continuous". W. Rudin has informed me that L. Carleson (unpublished) has shown that if $f$ is 3 times continuously differentiable, then the conclusion of the theorem holds for $J$ an interval.

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2. For \( i = 1, 2 \) we define \( \pi_i : [0, 1] \times [0, 1] \to T \) by \( \pi_i((x_1, x_2)) = \phi_i(x_i) \).

**Theorem 2.** Suppose that \( f : [0, 1] \to [0, 1] \) is strictly increasing and such that \( X = \{(x, f(x)) : x \in [0, 1]\} = A_1 \cup A_2 \) where \( m(\pi_i(A_i)) = 0 \) for \( i = 1 \) and \( 2 \). Then \( |\mu|(\phi_2(X)) = 0 \) for any \( \mu \perp A(U^2) \).

**Proof.** Let \( K \subset \phi_2(A_1) \) be compact. Define \( \tilde{K} = \phi_2^{-1}(K) \cap A_1 \). Since \( m(\pi_1(\tilde{K})) = 0 \), \( \pi_1(\tilde{K}) \) is an interpolation set for \( A(U) \). Suppose that \( g \in C(K) \). Since \( G \) defined by \( G(e^{ix_1}) = g((e^{ix_1}, e^{ix_2})) \), where \( (e^{ix_1}, e^{ix_2}) \) is the unique element of \( K \) having \( e^{ix_1} \) as its first component, is in \( C(\pi_1(\tilde{K})) \) and \( \pi_1(\tilde{K}) \) is an interpolation set for \( A(U) \), there is an \( H \in A(U) \) such that \( H = G \) on \( \pi_1(\tilde{K}) \). Define \( h \in A(U^2) \) by \( h(z_1, z_2) = H(z_1) \). Then \( h = g \) on \( K \). Thus \( K \) is an interpolation set for \( A(U^2) \). Thus if \( \mu \perp A(U^2) \), \( |\mu|(K) = 0 \). Hence \( |\tilde{\mu}|(\phi_2(A_1)) = 0 \) (where \( \tilde{\mu} \) is the completion of \( \mu \)). Similarly \( |\tilde{\mu}|(\phi_2(A_2)) = 0 \), and hence \( |\mu|(\phi_2(X)) = 0 \).

To show that continuity is not sufficient in Theorem 1, we apply Theorem 2 to a strictly increasing continuous \( f \) such that \( f'(x) = 0 \) for almost all \( x \) (\( dm \)). We choose \( A_2 = \{(x, f(x)) : f'(x) = 0\} \) and \( A_1 \) the remainder of the graph of \( f \). The equivalence between null sets and zero sets yields the desired result.

**REFERENCE**


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