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AN INEQUALITY FOR ANALYTIC FUNCTIONS

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ABSTRACT. If F denotes the boundary value of a function $f \in H^p$, $1 \leq p \leq \infty$, the infimum of the measure of $\{\theta \mid |F(\theta)| > A\}$ for given A, $0 < A < |f(0)|$, $\|f\|_{H^p} = 1$, is determined.

In this note we discuss an aspect of the boundary behaviour of certain functions analytic on the unit disc in terms of their values at the origin. Specifically, if $F(\theta)$ is the boundary value of a function $f \in H^p$, $1 \leq p \leq \infty$, and if a number A, $0 < A < |f(0)|$, is chosen, we are interested in *minimizing* the measure of $\{\theta \mid 0 \leq \theta \leq 2\pi \text{ and } |F(\theta)| > A\}$ over all $f \in H^p$ with $\|f\|_{H^p} = 1$. The result is that, for $p < \infty$, the infimum is the solution c of the equation

$$c \log[1 + 2\pi(1 - A^p)/cA^p] = 2\pi p \log(|f(0)|/A), \tag{1}$$

while if $p = \infty$, the infimum is $2\pi(1 - \log|f(0)|/\log A)$, which is the limit, as $p \to \infty$, of the solutions of (1).

The ingredients of the proof are Jensen's inequality and Jensen's formula which we state in the following forms.

1. **Jensen's inequality.** If μ is a positive measure on a measure space X with $\mu(X) = 1$, and if $f \in L^1(d\mu)$, then

$$\int_X \log|f|\, d\mu \leq \log \int_X |f|\, d\mu.$$

Equality holds if, and only if, f is an outer function [1, p. 62].

2. **Jensen's formula.** If $f \in H^1$ of the unit disc and $F(\theta) = \lim_{r \uparrow 1} f(re^{i\theta})$, then

$$\log|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|F(\theta)|\, d\theta.$$

Equality holds if, and only if, f is an outer function [1, p. 62].

We begin by showing, in Lemma 1, that for each pair of numbers A_0, A, $0 < A < A_0 < 1$, there is a step function h on $(0, 2\pi)$ satisfying $\int_0^{2\pi} |h| = 2\pi$ and $\int_0^{2\pi} \log|h| = 2\pi \log A_0$, and which takes on the value A and one other

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value $\bar{y}$. For H^1, the number $c = \text{measure}\{x \mid h(x) = \bar{y}\}$ will be the required solution of (1).

Lemma 1. *Given* $0 < A < A_0 < 1$, *there exist unique numbers* c *and* $\bar{y}$ *(depending on* A *and* A_0*) such that* $0 < c < 2\pi$, $1 < \bar{y}$ *and*

(2) $\quad 2\pi = (2\pi - c)A + c\bar{y}$ *and* $2\pi \log A_0 = (2\pi - c) \log A + c \log \bar{y}$.

Further, c *satisfies the equation*

$$c \log[1 + 2\pi(1 - A)/cA] = 2\pi \log(A/A_0). \tag{3}$$

Also, for fixed A, *the left-hand side of* (3) *is an increasing function of* c.

Proof. If $\bar{y}$ is the unique solution of

$$\frac{y - A}{1 - A} = \frac{\log y - \log A}{\log A_0 - \log A} \tag{4'}$$

for $y > A$, then $\bar{y} > 1$. Let

$$c = 2\pi(1 - A)/(\bar{y} - A). \tag{4''}$$

Since equations (4′) with $y = \bar{y}$ and (4″), together, are equivalent to equations (2), the pair $c, \bar{y}$ satisfies (2). Then from (4″), $0 < c < 2\pi$.

Equation (3) follows easily from (2) and a routine calculus argument proves the last statement of the lemma.

Lemma 2. *If* $0 < c < 2\pi$, $G \in L^1(0, 2\pi)$ *and* $|G| \leq A < 1$ *on a set* S *of measure* $2\pi - c$, *then*

$$(2\pi - c)A - \int_S |G| \leq (2\pi - c) \log A - \int_S \log |G|.$$

Proof. This follows immediately from the inequality $\log A - A \geq \log t - t$ for $0 < t \leq A < 1$.

Lemma 3. *Suppose* $0 < A < A_0 < 1$ *and* $c, \bar{y}$ *are the constants satisfying equations* (2) *of Lemma* 1. *Let* $F \in L^1(0, 2\pi)$ *with* $(1/2\pi) \int_0^{2\pi} |F| = 1$ *and* $(1/2\pi) \int_0^{2\pi} \log|F| = \log A_0$. *If* $|F| \leq A$ *on a set* S *of measure* $(2\pi - c)$, *then* $|F| = A$ *on* S *and* $|F| = \bar{y}$ *off* S.

Proof. Let $S' = [0, 2\pi] - S$. Then

$$\int_S |F| + \int_{S'} |F| = 2\pi \quad \text{and} \quad \int_S \log|F| + \int_{S'} \log|F| = 2\pi \log A_0.$$

From equations (2), c and $\bar{y}$ satisfy

$$2\pi = (2\pi - c)A + c\bar{y} \quad \text{and} \quad 2\pi \log A_0 = (2\pi - c) \log A + c \log \bar{y}.$$

Hence

$$\int_{S'} |F| = (2\pi - c)A + c\bar{y} - \int_S |F|$$

and

$$\int_{S'} \log|F| = (2\pi - c)\log A + c\log\bar{y} - \int_S \log|F|.$$

Let

(5) $$a = (2\pi - c)A - \int_S |F| \quad \text{and} \quad b = (2\pi - c)\log A - \int_S \log|F|.$$

Then

$$\int_{S'} |F| = a + c\bar{y} \quad \text{and} \quad \int_{S'} \log|F| = b + c\log\bar{y};$$

and also from Lemma 2, $0 \le a \le b$. Now, by Jensen's inequality

$$\int_{S'} \log|F|\frac{d\theta}{c} \le \log\int_{S'} |F|\frac{d\theta}{c},$$

so that $a/c + \log\bar{y} \le b/c + \log\bar{y} \le \log(a/c + \bar{y})$. Since $\bar{y} > 1$, it follows that $a = b = 0$.

Therefore,

$$\int_{S'} |F|\frac{d\theta}{c} = \bar{y} \quad \text{and} \quad \int_{S'} \log|F|\frac{d\theta}{c} = \log\bar{y},$$

and the uniqueness part of Jensen's inequality implies $|F| = \bar{y}$ on S'.

Also, since $a = b = 0$, equations (5) imply

$$\int_S |F| = (2\pi - c)A \quad \text{and} \quad \int_S \log|F| = (2\pi - c)\log A,$$

and so for the same reason, $|F| = A$ on S.

Notation. If S is a subset of $[0, 2\pi]$, $m(S)$ will denote the Lebesgue measure of S.

Theorem A. *Let $f \in H^1$ and $\|f\|_{H^1} = 1$. Suppose $0 < A < |f(0)|$ and let c be the solution of*

(6) $$c\log[1 + 2\pi(1 - A)/cA] = 2\pi\log(|f(0)|/A).$$

If $F(\theta) = \lim_{r\uparrow 1} f(re^{i\theta})$, then $m(\{\theta \mid |F(\theta)| > A\})$ is greater than c.

Proof. If f is an outer function, then $(1/2\pi)\int_0^{2\pi}|F| = 1$ and by Jensen's formula,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|F| = \log|f(0)|.$$

Therefore, by Lemma 3, $m(\{\theta \mid |F(\theta)| > A\})$ is bigger than c.

If f is not an outer function, write $f = BSg$, where B is the Blaschke product of the zeros of f, S is the singular part of f, and g is an outer function [1]. If $G(\theta) = \lim_{r\uparrow 1}|g(re^{i\theta})|$, then $|G(\theta)| = |F(\theta)|$ a.e., so that

$m(\{\theta|\ |G(\theta)| > A\}) = m(\{\theta|\ |F(\theta)| > A\})$. Also since g is an outer function we have $|g(0)| > |f(0)|$. By Lemma 1, we have that the solution c of $c\log[1 + 2\pi(1 - A)/cA] = 2\pi\log(|g(0)|/A)$ is bigger than the solution c of (6) concluding the proof.

We remark that the number c satisfying (6) is the best possible, since we can find $g \in H^1$ such that $|G|$ is arbitrarily close to the step function H: $H(\theta) = A$, $0 \le \theta < 2\pi - c$, $H(\theta) = \bar{y}$, $2\pi - c \le \theta < 2\pi$.

Now, if $f \in H^p$, $1 \le p < \infty$, then $f^p \in H^1$ and $m(\{\theta|\ |F(\theta)| > A\}) = m(\{\theta|\ |F(\theta)|^p > A^p\})$. This reduces to the H^1 case with f replaced by f^p and A by A^p. Hence,

Theorem B. *Let $f \in H^p$, $1 \le p < \infty$, and $\|f\|_{H^p} = 1$. If $0 < A < |f(0)|$, then $m(\{\theta|\ |F(\theta)| > A\})$ is greater than the solution c of*

$$c\log[1 + 2\pi(1 - A^p)/cA^p] = 2\pi p\log(|f(0)|/A)$$

and this is the best possible.

Finally, for H^∞ (or the disc algebra), if $f \in H^\infty$ and $\|f\|_\infty \le 1$, we have

$$2\pi\log|f(0)| \le \int_{|F|\le A}\log|F| + \int_{|F|>A}\log|F| \le \int_{|F|\le A}\log|F|$$

since $\log|F| < 0$.

Therefore, $2\pi\log|f(0)| \le \log A \cdot m(\{\theta|\ |F(\theta)| \le A\})$, or

$$2\pi\frac{\log|f(0)|}{\log A} \ge m(\{\theta|\ |F(\theta)| \le A\}) = 2\pi - m(\{\theta|\ |F(\theta)| > A\}),$$

or

$$m(\{\theta|\ |F(\theta)| > A\}) \ge 2\pi(1 - \log|f(0)|/\log A). \tag{7}$$

Again it is easy to see that this is the best possible.

Remarks. (I) The same methods apply, as well, to any positive measure μ on a measure space X for which Jensen's formula $\log|\int_X f\,d\mu| \le \int_X \log|f|\,d\mu$ holds. (See [1, pp. 54–57], for some examples.) For these cases, equations (1) and (7) are valid if 2π is replaced by $\mu(X)$.

(II) By considering the Poisson kernel, it follows from (I) that we can solve the same problem in the unit disc knowing f at some point z' other than 0. A particular result is that, if $f \in H^\infty$ of the unit disc, F is the boundary value of f, $|z'| < 1$ and $0 < A < |f(z')|$, then the *Lebesgue* measure of $\{\theta|\ |F(\theta)| \le A\}$ is less than or equal to

$$4 \tan^{-1}\left[\frac{1+|z'|}{1-|z'|} \tan \frac{\pi}{2} \frac{\log |f(z')|}{\log A}\right].$$

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