PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

HEREDITARY COMPLETENESS AND QUASI-REFLEXIVITY

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ABSTRACT. In this note, we prove that the countable direct sum of quasi-reflexive Banach spaces is hereditarily complete and hence the separated countable inductive limit of quasi-reflexive Banach spaces is complete.

The notion of quasi-reflexivity has been studied by Civin and Yood [1]. It is shown easily that the countable direct sum of reflexive Banach spaces is hereditarily complete (in fact even B-complete) while an example in Köthe [3] shows that the countable direct sum of Banach spaces is not in general hereditarily complete. In this note we prove the following

Theorem. *Let X_i, $i = 1, 2, 3, \cdots$, be a sequence of quasi-reflexive Banach spaces. Then $E = \bigoplus_{n=1}^{\infty} X_n$ with the usual direct sum topology is hereditarily complete.*

Proof. Let F be a closed subspace of (E, u) where u is the topology mentioned above. Then the dual of the quotient space E/F, i.e. F^0 with the relativised strong topology (which is also identical with $\beta(F^0, E/F)$), is a Fréchet space.

Let f be a linear functional which is continuous on the equicontinuous subsets of F^0. Since E/F has the Mackey topology, it follows that f is continuous on each $\sigma(F^0, E/F)$ compact convex subset of F^0 and hence clearly on all strongly compact subsets (continuity is with respect to $\sigma(F^0, E/F)$). This implies that $f \in E''/F^{00}$ since F^0 is a Fréchet space as mentioned above. Hence by Grothendieck's theorem [2] on completeness, $(E/F)^{\sim} \subseteq E''/F^{00}$(1).

Let E'' be given the strong topology with E''/F^{00} having the quotient topology. Next we show that the canonical image of E/F in E''/F^{00} is closed. If ψ denotes this canonical map, then $\psi(E/F) = (E + F^{00})/F^{00}$

Presented to the Society, September 4, 1973; received by the editors October 3, 1973.

AMS (MOS) subject classifications (1970). Primary 46B10.

Key words and phrases. Quasi-reflexive, barreled space, countable-codimensional subspace.

where we have identified E with its canonical image in E''. Now E is certainly closed in E'' and hence closed in $E + F^{00}$. By Valdivia [5], $E + F^{00}$ (being a subspace of countable codimension in the barreled space E'') is barreled and by the proposition of Saxon and Levin [4, p.–92], $E + F^{00} = E \oplus N$, where N is any algebraic complement of E in $E + F^{00}$ with the strongest locally convex topology. This at once implies that $E + F^{00}$ is complete and hence closed in E''. But this implies by (1) that

$$E/F = (E/F)^{\sim}, \quad \text{i.e. } E/F \text{ is complete.}$$

Note. The above Theorem has the following interesting

Corollary. *The separated countable inductive limit of quasi-reflexive Banach spaces is complete.*

The author wishes to express thanks to Professor A. C. Cochran and W. H. Summers for many helpful discussions.

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