

## HEREDITARY COMPLETENESS AND QUASI-REFLEXIVITY

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**ABSTRACT.** In this note, we prove that the countable direct sum of quasi-reflexive Banach spaces is hereditarily complete and hence the separated countable inductive limit of quasi-reflexive Banach spaces is complete.

The notion of quasi-reflexivity has been studied by Civin and Yood [1]. It is shown easily that the countable direct sum of reflexive Banach spaces is hereditarily complete (in fact even  $B$ -complete) while an example in Köthe [3] shows that the countable direct sum of Banach spaces is not in general hereditarily complete. In this note we prove the following

**Theorem.** Let  $X_i, i = 1, 2, 3, \dots$ , be a sequence of quasi-reflexive Banach spaces. Then  $E = \bigoplus_{n=1}^{\infty} X_n$  with the usual direct sum topology is hereditarily complete.

**Proof.** Let  $F$  be a closed subspace of  $(E, u)$  where  $u$  is the topology mentioned above. Then the dual of the quotient space  $E/F$ , i.e.  $F^0$  with the relativised strong topology (which is also identical with  $\beta(F^0, E/F)$ ), is a Fréchet space.

Let  $f$  be a linear functional which is continuous on the equicontinuous subsets of  $F^0$ . Since  $E/F$  has the Mackey topology, it follows that  $f$  is continuous on each  $\sigma(F^0, E/F)$  compact convex subset of  $F^0$  and hence clearly on all strongly compact subsets (continuity is with respect to  $\sigma(F^0, E/F)$ ). This implies that  $f \in E''/F^{00}$  since  $F^0$  is a Fréchet space as mentioned above. Hence by Grothendieck's theorem [2] on completeness,  $(E/F)^\sim \subseteq E''/F^{00}(1)$ .

Let  $E''$  be given the strong topology with  $E''/F^{00}$  having the quotient topology. Next we show that the canonical image of  $E/F$  in  $E''/F^{00}$  is closed. If  $\psi$  denotes this canonical map, then  $\psi(E/F) = (E + F^{00})/F^{00}$ .

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where we have identified  $E$  with its canonical image in  $E''$ . Now  $E$  is certainly closed in  $E''$  and hence closed in  $E + F^{00}$ . By Valdivia [5],  $E + F^{00}$  (being a subspace of countable codimension in the barreled space  $E''$ ) is barreled and by the proposition of Saxon and Levin [4, p.-92],  $E + F^{00} = E \oplus N$ , where  $N$  is any algebraic complement of  $E$  in  $E + F^{00}$  with the strongest locally convex topology. This at once implies that  $E + F^{00}$  is complete and hence closed in  $E''$ . But this implies by (1) that

$$E/F = (E/F)^\sim, \quad \text{i.e. } E/F \text{ is complete.}$$

*Note.* The above Theorem has the following interesting

**Corollary.** *The separated countable inductive limit of quasi-reflexive Banach spaces is complete.*

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