HEREDITARY COMPLETENESS AND QUASI-REFLEXIVITY

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ABSTRACT. In this note, we prove that the countable direct sum of quasi-reflexive Banach spaces is hereditarily complete and hence the separated countable inductive limit of quasi-reflexive Banach spaces is complete.

The notion of quasi-reflexivity has been studied by Civin and Yood [1]. It is shown easily that the countable direct sum of reflexive Banach spaces is hereditarily complete (in fact even $B$-complete) while an example in Köthe [3] shows that the countable direct sum of Banach spaces is not in general hereditarily complete. In this note we prove the following

**Theorem.** Let $X_i$, $i = 1, 2, 3, \ldots$, be a sequence of quasi-reflexive Banach spaces. Then $E = \bigoplus_{n=1}^{\infty} X_n$ with the usual direct sum topology is hereditarily complete.

**Proof.** Let $F$ be a closed subspace of $(E, u)$ where $u$ is the topology mentioned above. Then the dual of the quotient space $E/F$, i.e. $F^0$ with the relativised strong topology (which is also identical with $\beta(F^0, E/F)$), is a Fréchet space.

Let $f$ be a linear functional which is continuous on the equicontinuous subsets of $F^0$. Since $E/F$ has the Mackey topology, it follows that $f$ is continuous on each $\sigma(F^0, E/F)$ compact convex subset of $F^0$ and hence clearly on all strongly compact subsets (continuity is with respect to $\sigma(F^0, E/F)$). This implies that $f \in E''/F^{00}$ since $F^0$ is a Fréchet space as mentioned above. Hence by Grothendieck's theorem [2] on completeness, $(E/F)^{\sigma} \subset E''/F^{00}$.

Let $E''$ be given the strong topology with $E''/F^{00}$ having the quotient topology. Next we show that the canonical image of $E/F$ in $E''/F^{00}$ is closed. If $\psi$ denotes this canonical map, then $\psi(E/F) = (E + F^{00})/F^{00}$


Key words and phrases. Quasi-reflexive, barreled space, countable-codimensional subspace.

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where we have identified $E$ with its canonical image in $E''$. Now $E$ is certainly closed in $E''$ and hence closed in $E + F^{00}$. By Valdivia [5], $E + F^{00}$ (being a subspace of countable codimension in the barrelled space $E''$) is barrelled and by the proposition of Saxon and Levin [4, p. 92], $E + F^{00} = E \oplus N$, where $N$ is any algebraic complement of $E$ in $E + F^{00}$ with the strongest locally convex topology. This at once implies that $E + F^{00}$ is complete and hence closed in $E''$. But this implies by (1) that $E/F = (E/F)\sim$, i.e. $E/F$ is complete.

Note. The above Theorem has the following interesting Corollary. The separated countable inductive limit of quasi-reflexive Banach spaces is complete.

The author wishes to express thanks to Professor A. C. Cochran and W. H. Summers for many helpful discussions.

REFERENCES