PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Vol. 46, Number 2, November 1974

ON THE FAILURE OF THE FIRST PRINCIPLE OF SEPARATION FOR COANALYTIC SETS[1]

ASHOK MAITRA

ABSTRACT. In this note we present a new example of a pair of disjoint coanalytic sets which are not Borel separable, i.e., coanalytic sets D and H such that $D \cap H = \emptyset$ and such that there is no Borel set E for which $D \subseteq E$ and $E \cap H = \emptyset$.

1. There are in the literature several proofs of the existence of a pair of disjoint coanalytic sets which are not Borel separable [3, pp. 220, 260, 263], [4, p. 25], [5]. In this note we present yet another proof. We show that Blackwell's construction in [1] of a Borel set which does not admit a Borel uniformization yields explicitly a pair of disjoint coanalytic sets which are not Borel separable.

2. First, we briefly recall Blackwell's construction. Let U be the set of all finite sequences of positive integers of positive length. Let X be the power-set of U. Identify X with 2^U and endow X with the product of discrete topologies, so that X is a homeomorph of the Cantor set. With each $x \in X$, associate a game $G(x)$ between players α and β as follows: the players alternately choose positive integers, α choosing first, each choice being made with complete information about all previous choices. For any play $\omega = (n_1, n_2, \cdots)$, let $k(\omega)$ be the first i such that $(n_1, n_2, \cdots, n_i) \notin x$, and let $k(\omega) = \infty$ if $(n_1, n_2, \cdots, n_i) \in x$ for all i. A play ω is a win for α in $G(x)$ just in case $k(\omega)$ is even, it is a win for β if $k(\omega)$ is odd, and it is a draw if $k(\omega) = \infty$. In any game $G(x)$, the space Y_1 of strategies for α can be identified with the set N^N of (infinite) sequences of positive integers, which we equip with the product of discrete

Received by the editors August 15, 1973.

AMS (MOS) subject classifications (1970). Primary 04A15, 54H05; Secondary 02K30.

Key words and phrases. Analytic sets, coanalytic sets, Borel separable, first principle of separation.

[1]Research supported by NIH Research Grant No. GM-10525, National Institutes of Health, Public Health Service.

topologies. A similar remark applies to the space Y_2 of strategies for β. Let Y be the disjoint union of Y_1 and Y_2 and give Y the union topology, so that Y is a homeomorph of N^N. Finally, let B_1 be the set of $(x, y) \in X \times Y_1$ such that y ensures α at least a draw in $G(x)$, and let B_2 be the set of $(x, y) \in X \times Y_2$ such that y ensures β at least a draw in $G(x)$.

Then, Blackwell has proved that

(i) B_1, B_2 are Borel subsets of $X \times Y$ (indeed, B_1, B_2 are closed subsets of $X \times Y$);

(ii) $\pi(B_1 \cup B_2) = X$, where π denotes projection of $X \times Y$ to X;

(iii) with each $x \in X$, it is possible to associate $x' \in X$ and an ordered pair (A_1, A_2) of nonempty analytic subsets of X in such a way that

(a) $G(x')$ is a win for α if $x \in A_1 - A_2$,

(b) $G(x')$ is a win for β if $x \in A_2 - A_1$,

(c) the mapping $x \longrightarrow x'$ is Borel measurable, and

(d) for every ordered pair (A_1, A_2) of nonempty analytic subsets of X, there is $x \in X$ such that (A_1, A_2) is associated with x.

We shall say that "x codes (A_1, A_2)" in case (A_1, A_2) is associated with x.

3. We are now ready to state our example. Let $D = X - \pi(B_1)$, and let $H = X - \pi(B_2)$. In other words, D is the set of $x \in X$ such that $G(x)$ is a win for β, while H is the set of x's such that $G(x)$ is a win for α. We claim that D, H are disjoint coanalytic sets which are not Borel separable.

From (i) it follows that D and H are coanalytic. (ii) implies that $D \cap H = \emptyset$. Next we note that D, H are nonempty. To see this, let x be the singleton set whose only member is the ordered pair (1, 2). Plainly, $G(x)$ is a win for β, so that $x \in D$. A similar argument shows that $H \neq \emptyset$.

Now assume by way of contradiction that there is a Borel set $E \subseteq X$ such that $D \subseteq E$ and $E \cap H = \emptyset$. Plainly, E and $X - E$ are both nonempty. Let $W = \{x \in X : x' \in E\}$. By (iii)(c), W is Borel. We now assert that both W and $X - W$ are nonempty. To see, for instance, that $W \neq \emptyset$, choose $x_0 \in X$ and a nonempty analytic set $C \subseteq X$ such that x_0 codes (C, X) and $x_0 \notin C$ (we can do this; for, if not, then for each $z \in X$, z codes $(\{z\}, X)$, so that there are no codes left for the other pairs). Since $x_0 \in X - C$, it follows from (iii)(b) that $G(x_0')$ is a win for β, so $x_0' \in D$ and hence, $x_0 \in W$. One shows $X - W \neq \emptyset$ similarly.

Consequently, from (iii)(d), there is $x^* \in X$ such that x^* codes $(W, X - W)$. We now have:

$x^{*\prime} \in E \longrightarrow x^* \in W \longrightarrow G(x^{*\prime})$ is a win for $\alpha \longrightarrow x^{*\prime} \in H \longrightarrow x^{*\prime} \notin E$; and also,

$x^{*\prime} \notin E \to x^* \in X - W \to G(x^{*\prime})$ is a win for $\beta \to x^{*\prime} \in D \to x^{*\prime} \in E$, so that $x^{*\prime} \in E \leftrightarrow x^{*\prime} \notin E$, which yields the desired contradiction.

4. Finally, by using an argument due to Novikov [4, p. 25], one can deduce from the fact that D and H are not Borel separable that the set $B = B_1 \cup B_2$ does not admit a Borel uniformization. Indeed, suppose that S is a Borel uniformization of B. Let $T = \pi(S - B_1)$. Since π is continuous and one-one on S, T is Borel [2, p. 487]. Now verify that $D \subseteq T$ and $T \cap H = \emptyset$, which contradicts the fact that D and H are not Borel separable.

BIBLIOGRAPHY

1. D. Blackwell, *A Borel set not containing a graph*, Ann. Math. Statist. 39 (1968), 1345-1347. MR 37 #5025.

2. K. Kuratowski, *Topology*. Vol. 1, new ed., rev. and augmented, PWN, Warsaw; Academic Press, New York, 1966. MR 36 #840.

3. N. N. Luzin, *Leçons sur les ensembles analytiques et leurs applications*, Gauthier-Villars, Paris, 1930.

4. P. Novikov, *Sur les fonctions implicites mesurables B*, Fund. Math. 17 (1931), 8-25.

5. W. Sierpiński, *Sur deux complémentaries analytiques non séparables B*, Fund. Math. 17 (1931), 296-297.

DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Indian Statistical Institute, Calcutta, India