PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

NOTE ON BOOLEAN ULTRAPOWERS

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ABSTRACT. The paper gives a characterization of ultrafilters in Boolean algebras which make Boolean ultrapowers saturated. The description agrees with the definition of good ultrafilters if the Boolean algebra is atomic and is weaker if the Boolean algebra is atomless.

The aim of this note is to prove a result similar to a theorem of Keisler [1] which says that an ultrafilter is good iff powers reduced by it are saturated. Mansfield proved in [2] that if one defines the notion of a good ultrafilter in a Boolean algebra in a manner analogous to the ordinary definition, one can prove the implication from left to right by imitating Keisler's argument. At about the same time I observed that with this definition of goodness, it is not straightforward to prove the other implication, and I defined a notion which permits complete generalization of Keisler's theorem to Boolean ultrapowers. Whether the notion of goodness is truly weaker than Definition 4.1 of [2] is open.

We will define Boolean ultrapower of a structure $\mathbf{A}$ in a different form than the one used in [2]. Given a complete Boolean algebra, we say that $X \subseteq B - \{0\}$ is a partition if $x \cap y = 0$ whenever $x, y \in X$, $x \neq y$, and $\bigcup X = 1$ ($\bigcup X$ is the least upper bound of X in B, and $x \cap y$ is the meet of x and y in B). For a set A we define $A^{(B)}$ to be the set of all functions into A whose domain is a partition of B. Given a relation $R(\cdot, \cdot)$ on A, define, for $f, g \in A^{(B)}$,

$$\|R(f, g)\| = \bigcup \{x \cap y \mid x \in \operatorname{Dom} f,\ y \in \operatorname{Dom} g \text{ and } R(f(x), g(y))\}.$$

This definition can be extended to other relations and formulas in the natural way. Defining $H(f)$, for $f \in A^{(B)}$, by

$$H(f)(a) = \begin{cases} f^{-1}(a) & \text{if } a \in \operatorname{Range}(f), \\ 0 & \text{if not,} \end{cases}$$

it is seen that $H(f)$ is a member of Mansfield's $A^{(B)}$ and that H provides a

Received by the editors July 1, 1973.

AMS (MOS) subject classifications (1970). Primary 02H13, 02J05.

Key words and phrases. Boolean ultrapowers, saturated models, good ultrafilters.

B-isomorphism of these two definitions. Ours seems to be more analogous with the definition of ordinary ultrapower, while Mansfield's is related to Boolean-valued models for set theory.

Given partitions $P_0, \cdots, P_n$ in a Boolean algebra B, we denote the partition $\{x_0 \cap \cdots \cap x_n | x_i \in P_i \text{ for } i \leq n\} - \{0\}$ by $\bigwedge_{i=0}^n P_i$. We say that an element $b \in B$ is a part of a partition P in B if $b = \bigcup X$ for some $X \subseteq P$.

Definition. An ultrafilter D is said to be κ-good iff: for every $\lambda < \kappa$, for every function $f: S_\omega(\lambda) \to D$ for which there are partitions P_α for $\alpha < \lambda$ such that $f(s)$ is a part of $\bigwedge_{\alpha \in s} P_\alpha$ for every $s \in S_\omega(\lambda)$, there is a multiplicative function $g: S_\omega(\lambda) \to D$, $g \leq f$, and a partition P so that $g(s)$ is a part of $P \wedge \bigwedge_{\alpha \in s} P_\alpha$, and for every $p \in P$, the set $\{\alpha | g(\alpha) \cap p \neq 0\}$ is finite.

Remark. If $B = S(I)$ then this notion coincides with the usual notion of goodness. If we have a decreasing function $f: S_\omega(\lambda) \to D$, then $f(s)$ is a part of the partition $\{\{i\} | i \in I\}$. So there is a multiplicative function $g: S_\omega(\lambda) \to D$ refining f. Hence this definition is not weaker.

Let us now assume that D is a κ-good ultrafilter in the old sense and let $f: S_\omega(\lambda) \to D$. Let g be a multiplicative refinement of f. We can assume that $\{\alpha | i \in g(\{\alpha\})\}$ is finite for every $i \in I$; thus, taking P to be $\{\{i\} | i \in I\}$, we see that if $B = S(I)$, then the above notion of goodness agrees here with the ordinary one.

Theorem. *An ultrafilter D in the Boolean algebra B is κ-good iff $\mathbf{A}^{(B)}/D$ is κ-saturated for every structure $\mathbf{A}$.*

Proof. Let us assume that D is a κ-good ultrafilter, that $\lambda < \kappa$ and $\Phi = \{\phi_\alpha(x, \bar{f}_\alpha/D) | \alpha < \lambda\}$ is a set of formulas in the language of $L(\mathbf{A}^{(B)}/D)$ which is finitely satisfiable. ($\bar{f}_\alpha/D$ stands for a finite sequence of elements from $\mathbf{A}^{(B)}/D$.) Thus we have that for each $s \in S_\omega(\lambda)$,

$$f(s) = \left\| (\exists x) \bigwedge_{\alpha \in s} \phi_\alpha(x, \bar{f}_\alpha) \right\| \in D.$$

By [2, Theorem 1.1], $f(\{\alpha\})$ is a part of the partition $P_\alpha = \bigwedge D\bar{f}_\alpha$ (the meet of domains of the functions occurring in the sequence $\bar{f}_\alpha$). Since $\phi_s = (\exists v) \bigwedge_{\alpha \in s} \phi_\alpha(v, \bar{f}_\alpha)$ contains as constants only the functions from some sequence $\bar{f}_\alpha$ for some $\alpha \in s$, we see that $f(s)$ is a part of $\bigwedge_{\alpha \in s} P_\alpha$ for every $s \in S_\omega(\lambda)$. By the goodness of D there is a multiplicative $g: S_\omega(\lambda) \to D$, $g \leq f$, and a partition P which satisfies that for each $p \in P$, $s_p = \{\alpha | g(\alpha) \cap p \neq 0\}$ is finite. Let us consider the set $X_p = \{x | x = a \cap p \neq 0 \text{ for some } a \in \bigwedge_{\alpha \in s_p} P_\alpha\}$. It is clear that $\bigcup X_p = P$ and that it is a disjointed set. Thus $X = \bigcup_{p \in P} X_p$

is a partition in B. For every $x \in X$ let $t(x) = \{\alpha | x \leq g(\alpha)\}$. If $x \leq p$ (for every $x \in X$ such a $p \in P$ is unique) then $t(x) \subseteq s_p$. Because $g(s_p)$ is a part of $P \wedge \bigwedge_{\alpha \in s_p} P_\alpha$, we see that by multiplicativity of g, $x \leq g(t(x))$ for $x \in X$. Now $g(t(x)) \leq f(t(x))$. Because

$$x \neq 0 \quad \text{and} \quad x \leq f(t(x)) = \left\| (\exists v) \bigwedge_{\alpha \in t(x)} \phi_\alpha(v, \bar{f}_\alpha) \right\|,$$

we have that for $\alpha \in t(x)$ there is a $\bar{d}_\alpha$, a sequence of elements from the domains of $\bar{f}_\alpha$, so that $x \leq \bar{d}_\alpha$ for $\alpha \in t(x)$ and

$$\mathbf{A} \models (\exists v) \bigwedge_{\alpha \in t(x)} \phi_\alpha(v, \bar{f}_\alpha(\bar{d}_\alpha)).$$

We define $h(x)$ to be an element of $\mathbf{A}$ for which $\mathbf{A} \models \bigwedge_{\alpha \in t(x)} \phi_\alpha(h(x), \bar{f}_\alpha(\bar{d}_\alpha))$ holds. It is now clear that $\|\phi_\alpha(h, \bar{f}_\alpha)\| \geq x$ whenever $\alpha \in t(x)$. But

$$\bigcup \{x \mid \alpha \in t(x)\} = g(\{\alpha\}) \in D.$$

This shows that $\|\phi_\alpha(h, \bar{f}_\alpha)\| \in D$ for every $\alpha < \lambda$. For the other implication we need a lemma.

Lemma. *Let* $\{A_\alpha | \alpha < \kappa\}$ *be a disjointed set of nonempty sets, and let* f *be a function on* $S_\omega(\kappa)$ *such that* $f(s) \subseteq \Pi_{\alpha \in s} A_\alpha$, *and if* $s' \subseteq s$ *and* $\langle a_1, \cdots, a_m\rangle \in f(s)$, *then* $\langle a_i | i \in s'\rangle \in f(s')$. *Then there are functions* T_α $(\alpha < \kappa)$ *such that*:

(i) $DT_\alpha = A_\alpha$;

(ii) *if* $s = \{\alpha_1, \cdots, \alpha_n\} \in S_\omega(\kappa)$ *and* $\langle a_1, \cdots, a_n\rangle \in \Pi_{\alpha \in s} A_\alpha$, *then* $\langle a_1, \cdots, a_n\rangle \in f(s)$ *iff* $\bigcap_{i=1} T_{\alpha_i}(a_i) \neq 0$;

(iii) *if* $S \subset \kappa$, $|S| \geq \omega$, *then* $\bigcap_{\alpha \in S} T_\alpha(a_\alpha) = 0$ *for any* $a_\alpha \in A_\alpha$ $(\alpha \in S)$.

Proof. We can easily define (by induction) functions T^n on $\bigcup_{|s|=n} \Pi_{\alpha \in s} A_\alpha$ so that $\{T^n(\langle a_1, \cdots, a_n\rangle) | n < \omega, a_i \in A_{\alpha_i}\}$ is a disjointed set, and if $s = \{\alpha_i | i < n\}$ and $a_i \in A_{\alpha_i}$, then

$$T^n(\langle a_0, \cdots, a_{n-1}\rangle) \neq 0 \quad \text{iff } \langle a_0, \cdots, a_{n-1}\rangle \in f(s).$$

Let

$$T_\alpha(a) = \bigcup_{n<\omega} \left\{ T^n(\langle a_0, \cdots, a_{n-1}\rangle) | \langle a_0 \cdots a_{n-1}\rangle \in \bigcup_{|s|=n} \prod_{\beta \in s} A, \ a \text{ occurs in } \langle a_0, \cdots, a_{n-1}\rangle \right\}$$

for $a \in A_\alpha$. T_α's satisfy (i) by definition. To prove (ii) let $s = \{\alpha_0, \cdots, \alpha_{n-1}\}$, and let $\langle a_0, \cdots, a_{n-1}\rangle \in \Pi_{\alpha \in a} A_\alpha$. If $\langle a_0, \cdots, a_{n-1}\rangle \in f(s)$ then $T^n(\langle a_0, \cdots, a_{n-1}\rangle) \neq 0$ and $T^n(\langle a_0, \cdots, a_{n-1}\rangle) \subseteq T_{\alpha_i}(a_i)$ for every $i < n$

by definition. Hence $\bigcap_{i<n} T_{\alpha_i}(a_i) \neq 0$. On the other hand if $\bigcap_{i<n} T_{\alpha_i}(a_i) \neq 0$, then for some m, some $r = \{\beta_i | i < m\}$, and some $b_i \in A_{\beta_i}$ such that $\langle b_i | i < m\rangle \in f(r)$, we have that

$$0 \neq T^m(\langle b_0, \cdots, b_{m-1}\rangle) \subseteq \bigcap_{i<n} T_{\alpha_i}(a_i).$$

By the manner in which we have defined the T_α's, it follows that $a_i \in \{b_j \mid j < m\}$ for every $i < n$ and $s \subseteq r$. By the property of f we have that

$$\langle b_j \mid j \in s\rangle = \langle a_i \mid i \in s\rangle \in f(s).$$

For (iii) we let S be an infinite subset of κ and let $a_\alpha \in A_\alpha$ for every $\alpha \in S$. If $\bigcap_{\alpha \in S} T_\alpha(a_\alpha) \neq 0$, then for some $m < \omega$, some $s = \{\alpha_i | i < m\}$, and some $b_i \in A_{\alpha_i}$ for $i < m$, we have

$$0 \neq T^m(\langle b_0, \cdots, b_{m-1}\rangle) \subseteq \bigcap_{\alpha \in S} T_\alpha(a_\alpha).$$

But this is a contradiction since this would mean that $\{a_\alpha | \alpha \in S\} \subseteq \{b_i | i < m\}$.

We now turn back to the proof of the Theorem. Let D be an ultrafilter in B having the property that $\mathbf{A}^{(B)}/D$ is κ-saturated for every $\mathbf{A}$. Let $f: S_\omega(\lambda) \to D$ be a decreasing function, and let P_α for $\alpha < \lambda$ be partitions in B such that $f(s)$ is a part of $\bigwedge_{\alpha \in s} P_\alpha$ for every $s \in S_\omega(\lambda)$. Using the just proven lemma, we can find functions T_α defined on P_α so that

(1) if $s = \{\alpha_0, \cdots, \alpha_{n-1}\} \in S_\omega(\lambda)$ and $a_i \in P_{\alpha_i}$, then $\bigcap_{i<n} a_i \neq 0$ and $\bigcap_{i<n} a_i \leq f(s)$ iff $\bigcap_{i<n} T_{\alpha_i}(a_i) \neq 0$, and

(2) for every $S \subseteq \kappa$, $|S| = \omega$, if $a_\alpha \in P_\alpha$ for every $\alpha \in S$, then $\bigcap_{\alpha \in S} T_\alpha(a_\alpha) = 0$.

It is quite clear that the functions T_α can be chosen so that $\mathrm{Ran}(T) \subseteq S(I)$ for some I. Let us consider the structure $\mathbf{A} = \langle S(I), \subseteq, 0\rangle$, where 0 denotes the empty set. The set of formulas $v \subseteq T_\alpha/D \wedge v \neq 0$ (these formulas are from the language $L(\mathbf{A}^{(B)}/D)$ since T_α's are elements of $S(I)^{(B)}$) is clearly finitely satisfiable in $\mathbf{A}^{(B)}/D$. In fact

$$\left\| (\exists v)\left[\bigwedge_{\alpha \in s} v \neq 0 \wedge v \subseteq T_\alpha\right] \right\| = \bigcup \left\{ \bigcap_{\alpha \in s} p_\alpha \,\Big|\, \bigcap_{\alpha \in s} T_\alpha(p_\alpha) \neq 0 \right\} = f(s) \in D$$

as follows from (1). Hence, by κ-saturatedness of $\mathbf{A}^{(B)}/D$, there is a $T \in S(I)^{(B)}$ such that $\|T \neq 0\| \wedge \|T \subseteq T_\alpha\| \in D$ for every $\alpha < \lambda$. Let $P = \mathrm{Dom}(T)$ and let

$$g(s) = \|T \neq 0\| \wedge \bigcap_{\alpha \in s} \|T \subseteq T_\alpha\|.$$

It is quite clear that g is a multiplicative function from $S_\omega(\lambda)$ into D. By Theorem 1.1 we have immediately that $g(s)$ is a part of the partition $P \wedge \bigwedge_{\alpha\in s} P_\alpha$. If $0 \neq p \wedge \bigcap_{\alpha\in s} p_\alpha \leq g(s)$ for some $p_\alpha \in P_\alpha$ $(\alpha \in s)$, then $0 \neq T(p) \subseteq \bigcap_{\alpha\in s} T_\alpha(p_\alpha)$; hence by (1), $\bigcap_{\alpha\in s} p_\alpha \leq f(s)$. From this it follows that $g \leq f$. It remains to prove that $\{\alpha | g(\alpha) \cap p \neq 0\}$ is finite for every $p \in P$. Indeed, if for infinitely many α's $\|T = 0\| \wedge \|T \subseteq T_\alpha\| \wedge p \neq 0$, then for infinitely many α's there is a $p_\alpha \in P_\alpha$ so that $T(p) \subseteq T_\alpha(p_\alpha)$ and $T(p) \neq 0$. This is impossible by (2). The proof of the Theorem is completed.

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