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LATTICE POINTS AND LIE GROUPS. III

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ABSTRACT. If a compact, simply connected, semisimple Lie group is considered as a Riemannian manifold with metric arising from the negative of the Killing form it is shown that its volume is

$$(4\pi)^{\dim G/2}\Gamma(\dim G/2 + 1)(1/|w|)\int_{|\Lambda|\leq 1} f^2(\Lambda)\,d\Lambda.$$

1. **Introduction.** Let (M, g) be a compact Riemannian manifold, M, with metric g and Laplacian Δ. If $0 = \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of $-\Delta$ it is well known from the results of Minakshisundaram and Pleijel [3] that $Z(t) = \Sigma_1^\infty e^{-\lambda_n t}$, the zeta-function of M, is asymptotic to (Riemannian volume of $M)/(4\pi t)^{\dim M/2}$ as $t \downarrow 0$. In this paper we will make specific calculations of the volume for the case of a compact, connected, simply connected, semisimple Lie group G. We denote by e the identity element of the group G and by $\mathfrak{g}$ the Lie algebra of G. The metric g will act on the tangent space at e, i.e. $\mathfrak{g}$, as the negative of the Killing form and g will be extended to the entire group by left translation. Before proceeding further the author wishes to thank Burton Randol and Joseph Wolf for their observations.

2. $Z(t)$. Having fixed the metric g, we wish to compute $Z(t)$. If we view the elements of the universal enveloping algebra of $\mathfrak{g}$ as left-invariant differential operators, the negative of the Casimir operator, C, becomes the Laplacian Δ of (G, g). We now mention a few properties of C. For any representation π of G (or equivalently $\mathfrak{g}$) $\pi(C)$ acts diagonally. Using the bijection between the irreducible representations of G and dominant integral weights, λ, in the dual of a Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbf{C}$, we will write π_λ for the representation with dominant weight λ. Then

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$\pi_\lambda(C) = (|\Lambda|^2 - |\delta|^2)\mathrm{Id}$ where $\Lambda = \lambda + \delta$, $\delta = \frac{1}{2}$ the sum of the positive roots of $\mathfrak{g}_C$ and norms are taken with respect to the Killing form. Now using the Peter-Weyl theorem and the Weyl character formula as in [1]

$$Z(t) = \sum_{\Lambda > 0} f^2(\Lambda) \exp\{-(|\Lambda|^2 - |\delta|^2)t\} = \frac{1}{|w|} \sum_{\Lambda} f^2(\Lambda) \exp\{-(|\Lambda|^2 - |\delta|^2)t\}$$

where $f(\Lambda) = \Pi_{\alpha>0}(\Lambda, \alpha)/\Pi_{\alpha>0}(\delta, \alpha)$ and $|w|$ is the order of the Weyl group.

3. **Calculation of the volume.** To analyze $Z(t)$ we use an Abelian theorem. We define a measure $dm(\lambda)$ on $(0, \infty)$ as a sum of Dirac delta functions

$$dm(\lambda) = \sum_{y = |\Lambda|^2 - |\delta|^2} c_y \delta_y(\lambda) \quad \text{where } c_y = \frac{1}{|w|} \sum_{\Lambda : |\Lambda|^2 - |\delta|^2 = y} f^2(\Lambda).$$

Then $Z(t) = \int_0^\infty e^{-\lambda t}\, dm(\lambda)$, the Laplace transform of $dm(\lambda)$. If $m(x) = \int_0^x dm(\lambda)$, then from [1]

$$m(x) \sim x^{\dim G/2} \frac{1}{|w|} \int_{|\Lambda| \le 1} f^2(\Lambda)\, d\Lambda,$$

where $d\Lambda$ is Haar measure on the real span of the roots in the dual of $\mathfrak{h}$ normalized such that the volume of the parallelepiped spanned by the fundamental weights equals 1.

Using an Abelian theorem [2, p. 420]

$$Z(t) \sim \Gamma(\dim G/2 + 1)\, m(1/t) \qquad (t \downarrow 0)$$

$$\sim \Gamma(\dim G/2 + 1) \frac{1}{t^{\dim G/2}} \frac{1}{|w|} \int_{|\Lambda| \le 1} f^2(\Lambda)\, d\Lambda.$$

Using the results of [3]

$$Z(t) \sim (\text{Volume } G)/(4\pi t)^{\dim G/2}.$$

Thus

$$\text{Volume } G = (4\pi)^{\dim G/2}\Gamma(\dim G/2 + 1) \frac{1}{|w|} \int_{|\Lambda| \le 1} f^2(\Lambda)\, d\Lambda.$$

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