PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

LATTICE POINTS AND LIE GROUPS. III

ROBERT S. CAHN

ABSTRACT. If a compact, simply connected, semisimple Lie group is considered as a Riemannian manifold with metric arising from the negative of the Killing form it is shown that its volume is

$$(4\pi)^{\dim G/2}\Gamma(\dim G/2 + 1)(1/|w|) \int_{|\Lambda|\le 1} f^2(\Lambda)\, d\Lambda .$$

1. Introduction. Let (M, g) be a compact Riemannian manifold, M, with metric g and Laplacian Δ. If $0 = \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of $-\Delta$ it is well known from the results of Minakshisundaram and Pleijel [3] that $Z(t) = \Sigma_1^\infty e^{-\lambda_n t}$, the zeta-function of M, is asymptotic to (Riemannian volume of $M)/(4\pi t)^{\dim M/2}$ as $t \downarrow 0$. In this paper we will make specific calculations of the volume for the case of a compact, connected, simply connected, semisimple Lie group G. We denote by e the identity element of the group G and by $\mathfrak{g}$ the Lie algebra of G. The metric g will act on the tangent space at e, i.e. $\mathfrak{g}$, as the negative of the Killing form and g will be extended to the entire group by left translation. Before proceeding further the author wishes to thank Burton Randol and Joseph Wolf for their observations.

2. $Z(t)$. Having fixed the metric g, we wish to compute $Z(t)$. If we view the elements of the universal enveloping algebra of $\mathfrak{g}$ as left-invariant differential operators, the negative of the Casimir operator, C, becomes the Laplacian Δ of (G, g). We now mention a few properties of C. For any representation π of G (or equivalently $\mathfrak{g}$) $\pi(C)$ acts diagonally. Using the bijection between the irreducible representations of G and dominant integral weights, λ, in the dual of a Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbf{C}$, we will write π_λ for the representation with dominant weight λ. Then

Received by the editors July 27, 1973 and, in revised form, September 12, 1973 and October 12, 1973.

AMS(MOS) subject classifications (1970). Primary 35P20, 58G05.

Key words and phrases. Compact semisimple group, Casimir operator, Laplacian, zeta function.

$\pi_\lambda(C) = (|\Lambda|^2 - |\delta|^2)\mathrm{Id}$ where $\Lambda = \lambda + \delta$, $\delta = \frac{1}{2}$ the sum of the positive roots of $\mathfrak{g}_C$ and norms are taken with respect to the Killing form. Now using the Peter-Weyl theorem and the Weyl character formula as in [1]

$$Z(t) = \sum_{\Lambda>0} f^2(\Lambda)\exp\{-(|\Lambda|^2 - |\delta|^2)t\} = \frac{1}{|w|}\sum_{\Lambda} f^2(\Lambda)\exp\{-(|\Lambda|^2 - |\delta|^2)t\}$$

where $f(\Lambda) = \Pi_{\alpha>0}(\Lambda, \alpha)/\Pi_{\alpha>0}(\delta, \alpha)$ and $|w|$ is the order of the Weyl group.

3. Calculation of the volume. To analyze $Z(t)$ we use an Abelian theorem. We define a measure $dm(\lambda)$ on $(0, \infty)$ as a sum of Dirac delta functions

$$dm(\lambda) = \sum_{y=|\Lambda|^2-|\delta|^2} c_y\delta_y(\lambda) \quad \text{where } c_y = \frac{1}{|w|}\sum_{\Lambda:\,|\Lambda|^2-|\delta|^2=y} f^2(\Lambda).$$

Then $Z(t) = \int_0^\infty e^{-\lambda}\, dm(\lambda)$, the Laplace transform of $dm(\lambda)$. If $m(x) = \int_0^x dm(\lambda)$, then from [1]

$$m(x) \sim x^{\dim G/2}\frac{1}{|w|}\int_{|\Lambda|\le 1} f^2(\Lambda)\, d\Lambda,$$

where $d\Lambda$ is Haar measure on the real span of the roots in the dual of $\mathfrak{h}$ normalized such that the volume of the parallelepiped spanned by the fundamental weights equals 1.

Using an Abelian theorem [2, p. 420]

$$Z(t) \sim \Gamma(\dim G/2 + 1)\, m(1/t) \qquad (t \downarrow 0)$$

$$\sim \Gamma(\dim G/2 + 1)\frac{1}{t^{\dim G/2}}\frac{1}{|w|}\int_{|\Lambda|\le 1} f^2(\Lambda)\, d\Lambda.$$

Using the results of [3]

$$Z(t) \sim (\text{Volume } G)/(4\pi t)^{\dim G/2}.$$

Thus

$$\text{Volume } G = (4\pi)^{\dim G/2}\Gamma(\dim G/2 + 1)\frac{1}{|w|}\int_{|\Lambda|\le 1} f^2(\Lambda)\, d\Lambda.$$

BIBLIOGRAPHY

1. R. S. Cahn, *Lattice points and Lie groups*. II, Trans. Amer. Math. Soc. **183** (1973), 131–137.

2. W. Feller, *An introduction to probability and its applications*. Vol. II, Wiley, New York, 1966. MR 35 #1048.

3. S. Minakshisundaram and Å. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canad. J. Math. 1 (1949), 242–256. MR **11**, 108.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124