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TOPOLOGIES ON THE QUASI-SPECTRUM OF A C^*-ALGEBRA

WAI-MEE CHING [1]

ABSTRACT. We study two topologies, the Gelfand topology and the Jacobson topology, on the quasi-spectrum of a C^*-algebra. We prove that the Gelfand topology is finer than the Jacobson topology.

1. **Introduction.** The quasi-spectrum $\hat{A}$ of a C^*-algebra A is the set of all quasi-equivalence classes of factor representations of A. For a separable C^*-algebra A, we can put a Borel structure of Mackey on $\hat{A}$. This has been studied in detail, for example, in Dixmier [1, §7]. In this note, we study the topologies on $\hat{A}$ and prove that $\hat{A}$ is locally compact in the hull-kernel topology. As in the case of studying the spectrum of A, there are two ways to introduce a topology on $\hat{A}$: the quotient topology derived from the weak*-topology of $F(A)$, the set of all factor states of A, and the inverse image of the hull-kernel topology on the space X of all the kernels of factor representations of A. We prove that the first topology is finer than the second topology on $\hat{A}$. The parallel case for the topology on the spectrum of a C^*-algebra A, i.e., the set of unitary equivalence classes of all irreducible representations of A, has been treated in Dixmier [1, §3.4].

2. **The quasi-spectrum and the algebraic spectrum.** Let A be a C^*-algebra. A *state* f of A is a positive linear functional on A with $\|f\| = 1$. If A has an identity I, then a positive linear functional f on A is a state if and only if $f(I) = 1$. In case A is without identity, let A_I be the C^*-algebra derived from A by adjoining an identity I to A. Then, every state f of A has a unique extension to A_I, which we denote again by f, by defining $f(I) = 1$. If f is a state of A_I to start with, we also denote by f its restriction to A. We always denote by π_f the representation of A on the Hilbert space H_f associated with a state f of A by the standard Gelfand-Segal construction. A state f of a C^*-algebra A is called a *factor state*

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if $\pi_f(A)''$ is a factor, where S' denotes the commutant of S in $B(H_f)$, the set of bounded linear operators on H_f. Let $F(A)$ be the set of all factor states of a C^*-algebra A.

Definition. Two representations π and ρ of a C^*-algebra A on Hilbert spaces H and K are *quasi-equivalent* (resp. *algebraically equivalent*) if there exists an algebra isomorphism ϕ preserving * from $\pi(A)''$ onto $\rho(A)''$ (resp. from $\pi(A)$ onto $\rho(A)$) such that $\phi(\pi(x)) = \rho(x)$ for all $x \in A$. The *quasi-spectrum* $\hat{A}$ (resp. *algebraic spectrum* X) of a C^*-algebra A is the set of all quasi-equivalence (resp. algebraic equivalence) classes of factor representations of A.

Remark 1. Algebraic equivalence is a weaker notion than quasi-equivalence. Two quasi-equivalent representations of a C^*-algebra are obviously algebraically equivalent. But the converse is not true as the following example shows. Let A be a factor of type II_1. Since A is simple (von Neumann [3, p. 88(V)]), any nonzero representation of A is an isomorphism of A, hence algebraically equivalent to the identity representation. Now, let f be a pure state of A. π_f is irreducible, a factor representation of type I, not quasi-equivalent to the identity representation of A.

Remark 2. Two representations π and ρ of a C^*-algebra A are algebraically equivalent if and only if $\ker \pi = \ker \rho$. Therefore, we can regard the algebraic spectrum X of A as the set of all ideals which are kernels of factor representations of A.

3. **Topologies on the quasi-spectrum.** We have the following quotient maps:

$$F(A) \xrightarrow{q} \hat{A} \xrightarrow{p} X, \qquad f \mapsto [\pi_f] \mapsto \ker \pi_f,$$

where $[\pi_f]$ is the quasi-equivalence class of π_f. We equip X with the hull-kernel topology as we usually do for the structure space $\mathrm{Prim}(A)$, the set of all primitive ideals in A. A set S in X is closed if and only if $J_0 \supset \bigcap_{J \in S} J$ implies $J_0 \in S$. As in the case for the spectrum of A, there are naturally two topologies on $\hat{A}$. One is the topology pulled back from X. The other is the quotient topology relative to q and the weak*-topology of $F(A)$. We call the first topology J, Jacobson topology, and the second topology G, Gelfand topology. More precisely, a subset O of $\hat{A}$ is J-open if and only if O is the inverse image of a set open in X in the hull-kernel topology, and O is G-open if and only if $q^{-1}(O)$ is open in the weak*-topology. The Gelfand topology on $\hat{A}$ is finer than the Jacobson topology on $\hat{A}$. In order to prove this we first note that a state f of a C^*-algebra A is uniquely determined by its kernel $\ker f$, a linear subspace of A of codimension one. f can be identified with the quotient map $A \longrightarrow A/\ker f$. By an abuse of language, we shall call a subset S of $F(A)$ hull-kernel closed if $f \in F(A)$, $\ker f \supset \bigcap_{g \in S} \ker g$ implies $f \in S$.

Theorem. *The identity map from* $(\hat{A}, J)$ *to* $(\hat{A}, G)$ *is open.*

Proof. We first show that a hull-kernel closed set in $F(A)$ is closed in the weak*-topology of $F(A)$. Let S be a subset of $F(A)$. Let $\{f_i\}_{i \in I}$ be a net of states in S which converges to a state f, i.e., f is in the weak*-closure of S. For any $x \in \bigcap_{i \in I} \ker f_i$, we certainly have $f(x) = 0$. Hence,

$$\ker f \supset \bigcap_{i \in I} \ker f_i \supset \bigcap_{g \in S} \ker g.$$

Therefore, f is in the hull-kernel closure of S. Thus, S is hull-kernel closed implies S is weak*-closed. This proves the statement.

Now, let K be a J-closed subset of $\hat{A}$. We claim that $V = q^{-1}(K)$ is hull-kernel closed in $F(A)$. Let f be in the hull-kernel closure of V. Then,

$$\ker f \supset \bigcap_{g \in V} \ker g \supset \bigcap_{g \in V} \ker \pi_g.$$

By Corollary 2.4.10 of Dixmier [1], $\ker \pi_f$ is the largest norm-closed two-sided ideal inside $\ker f$. Hence, $\ker \pi_f$ contains the two-sided ideal $\bigcap_{g \in V} \ker \pi_g$. This means that $p(\pi_f)$ is in the closure of $p(K)$, which is equal to $p(K)$ since K is J-closed. Hence, $[\pi_f] \in K$, and $f \in V$. Therefore, V is hull-kernel closed in $F(A)$, and by the above statement, weak*-closed in $F(A)$. This shows that K is G-closed and completes the proof.

Remark. The same two topologies on the spectrum of a C^*-algebra A, the quotient topology from the weak*-topology of $P(A)$, the set of all pure states on A, and the topology induced from the hull-kernel topology of $\mathrm{Prim}(A)$, always coincide (Dixmier [1, Theorem 3.4.11]).

Proposition. *The algebraic spectrum* X *of a* C^*-*algebra* A *is locally compact.*

Proof. By exactly the same argument as that in [1, Proposition 3.3.7], we can prove that for an element a in A, and a positive number $r > 0$, the set K_r of all $\ker \pi$ in X such that $\|\pi(a)\| \geq r$ is compact. We only need to change $\pi \in \hat{A}$ to $\ker \pi \in X$, and note that by the same argument as in [1, Proposition 3.3.2], the mapping $N: \ker \pi \longrightarrow \|\pi(a)\|$ is lower semicontinuous on X and attains its supremum $\|a\|$ on X since X contains the spectrum of A.

Corollary. *(A, J) is locally compact.*

Proof. Let $[\pi] \in \hat{A}$. Let U be a compact neighborhood of $\ker \pi$. Then, $p^{-1}(U)$ is a compact neighborhood of $[\pi]$.

Remark. Kaplansky [2] has proved a similar result that the structure space Prim(A), the set of all kernels of irreducible representations of a C^*-algebra A, is locally compact.

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DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NEW YORK 10458