

## TOPOLOGIES ON THE QUASI-SPECTRUM OF A $C^*$ -ALGEBRA

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**ABSTRACT.** We study two topologies, the Gelfand topology and the Jacobson topology, on the quasi-spectrum of a  $C^*$ -algebra. We prove that the Gelfand topology is finer than the Jacobson topology.

**1. Introduction.** The quasi-spectrum  $\hat{A}$  of a  $C^*$ -algebra  $A$  is the set of all quasi-equivalence classes of factor representations of  $A$ . For a separable  $C^*$ -algebra  $A$ , we can put a Borel structure of Mackey on  $\hat{A}$ . This has been studied in detail, for example, in Dixmier [1, §7]. In this note, we study the topologies on  $\hat{A}$  and prove that  $\hat{A}$  is locally compact in the hull-kernel topology. As in the case of studying the spectrum of  $A$ , there are two ways to introduce a topology on  $\hat{A}$ : the quotient topology derived from the weak\*-topology of  $F(A)$ , the set of all factor states of  $A$ , and the inverse image of the hull-kernel topology on the space  $X$  of all the kernels of factor representations of  $A$ . We prove that the first topology is finer than the second topology on  $\hat{A}$ . The parallel case for the topology on the spectrum of a  $C^*$ -algebra  $A$ , i.e., the set of unitary equivalence classes of all irreducible representations of  $A$ , has been treated in Dixmier [1, §3.4].

**2. The quasi-spectrum and the algebraic spectrum.** Let  $A$  be a  $C^*$ -algebra. A state  $f$  of  $A$  is a positive linear functional on  $A$  with  $\|f\| = 1$ . If  $A$  has an identity  $I$ , then a positive linear functional  $f$  on  $A$  is a state if and only if  $f(I) = 1$ . In case  $A$  is without identity, let  $A_I$  be the  $C^*$ -algebra derived from  $A$  by adjoining an identity  $I$  to  $A$ . Then, every state  $f$  of  $A$  has a unique extension to  $A_I$ , which we denote again by  $f$ , by defining  $f(I) = 1$ . If  $f$  is a state of  $A_I$  to start with, we also denote by  $f$  its restriction to  $A$ . We always denote by  $\pi_f$  the representation of  $A$  on the Hilbert space  $H_f$  associated with a state  $f$  of  $A$  by the standard Gelfand-Segal construction. A state  $f$  of a  $C^*$ -algebra  $A$  is called a *factor state*

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if  $\pi_f(A)''$  is a factor, where  $S'$  denotes the commutant of  $S$  in  $B(H_f)$ , the set of bounded linear operators on  $H_f$ . Let  $F(A)$  be the set of all factor states of a  $C^*$ -algebra  $A$ .

**Definition.** Two representations  $\pi$  and  $\rho$  of a  $C^*$ -algebra  $A$  on Hilbert spaces  $H$  and  $K$  are *quasi-equivalent* (resp. *algebraically equivalent*) if there exists an algebra isomorphism  $\phi$  preserving  $*$  from  $\pi(A)''$  onto  $\rho(A)''$  (resp. from  $\pi(A)$  onto  $\rho(A)$ ) such that  $\phi(\pi(x)) = \rho(x)$  for all  $x \in A$ . The *quasi-spectrum*  $\hat{A}$  (resp. *algebraic spectrum*  $X$ ) of a  $C^*$ -algebra  $A$  is the set of all quasi-equivalence (resp. algebraic equivalence) classes of factor representations of  $A$ .

**Remark 1.** Algebraic equivalence is a weaker notion than quasi-equivalence. Two quasi-equivalent representations of a  $C^*$ -algebra are obviously algebraically equivalent. But the converse is not true as the following example shows. Let  $A$  be a factor of type  $II_1$ . Since  $A$  is simple (von Neumann [3, p. 88(V)]), any nonzero representation of  $A$  is an isomorphism of  $A$ , hence algebraically equivalent to the identity representation. Now, let  $f$  be a pure state of  $A$ .  $\pi_f$  is irreducible, a factor representation of type I, not quasi-equivalent to the identity representation of  $A$ .

**Remark 2.** Two representations  $\pi$  and  $\rho$  of a  $C^*$ -algebra  $A$  are algebraically equivalent if and only if  $\ker \pi = \ker \rho$ . Therefore, we can regard the algebraic spectrum  $X$  of  $A$  as the set of all ideals which are kernels of factor representations of  $A$ .

3. **Topologies on the quasi-spectrum.** We have the following quotient maps:

$$F(A) \xrightarrow{q} \hat{A} \xrightarrow{p} X, \quad f \mapsto [\pi_f] \mapsto \ker \pi_f,$$

where  $[\pi_f]$  is the quasi-equivalence class of  $\pi_f$ . We equip  $X$  with the hull-kernel topology as we usually do for the structure space  $\text{Prim}(A)$ , the set of all primitive ideals in  $A$ . A set  $S$  in  $X$  is closed if and only if  $J_0 \supset \bigcap_{J \in S} J$  implies  $J_0 \in S$ . As in the case for the spectrum of  $A$ , there are naturally two topologies on  $\hat{A}$ . One is the topology pulled back from  $X$ . The other is the quotient topology relative to  $q$  and the weak  $*$ -topology of  $F(A)$ . We call the first topology  $J$ , Jacobson topology, and the second topology  $G$ , Gelfand topology. More precisely, a subset  $O$  of  $\hat{A}$  is  $J$ -open if and only if  $O$  is the inverse image of a set open in  $X$  in the hull-kernel topology, and  $O$  is  $G$ -open if and only if  $q^{-1}(O)$  is open in the weak  $*$ -topology. The Gelfand topology on  $\hat{A}$  is finer than the Jacobson topology on  $\hat{A}$ . In order to prove

this we first note that a state  $f$  of a  $C^*$ -algebra  $A$  is uniquely determined by its kernel  $\ker f$ , a linear subspace of  $A$  of codimension one.  $f$  can be identified with the quotient map  $A \rightarrow A/\ker f$ . By an abuse of language, we shall call a subset  $S$  of  $F(A)$  hull-kernel closed if  $f \in F(A)$ ,  $\ker f \supset \bigcap_{g \in S} \ker g$  implies  $f \in S$ .

**Theorem.** *The identity map from  $(\hat{A}, J)$  to  $(\hat{A}, G)$  is open.*

**Proof.** We first show that a hull-kernel closed set in  $F(A)$  is closed in the weak  $^*$ -topology of  $F(A)$ . Let  $S$  be a subset of  $F(A)$ . Let  $\{f_i\}_{i \in I}$  be a net of states in  $S$  which converges to a state  $f$ , i.e.,  $f$  is in the weak  $^*$ -closure of  $S$ . For any  $x \in \bigcap_{i \in I} \ker f_i$ , we certainly have  $f(x) = 0$ . Hence,

$$\ker f \supset \bigcap_{i \in I} \ker f_i \supset \bigcap_{g \in S} \ker g.$$

Therefore,  $f$  is in the hull-kernel closure of  $S$ . Thus,  $S$  is hull-kernel closed implies  $S$  is weak  $^*$ -closed. This proves the statement.

Now, let  $K$  be a  $J$ -closed subset of  $\hat{A}$ . We claim that  $V = q^{-1}(K)$  is hull-kernel closed in  $F(A)$ . Let  $f$  be in the hull-kernel closure of  $V$ . Then,

$$\ker f \supset \bigcap_{g \in V} \ker g \supset \bigcap_{g \in V} \ker \pi_g.$$

By Corollary 2.4.10 of Dixmier [1],  $\ker \pi_f$  is the largest norm-closed two-sided ideal inside  $\ker f$ . Hence,  $\ker \pi_f$  contains the two-sided ideal  $\bigcap_{g \in V} \ker \pi_g$ . This means that  $p(\pi_f)$  is in the closure of  $p(K)$ , which is equal to  $p(K)$  since  $K$  is  $J$ -closed. Hence,  $[\pi_f] \in K$ , and  $f \in V$ . Therefore,  $V$  is hull-kernel closed in  $F(A)$ , and by the above statement, weak  $^*$ -closed in  $F(A)$ . This shows that  $K$  is  $G$ -closed and completes the proof.

**Remark.** The same two topologies on the spectrum of a  $C^*$ -algebra  $A$ , the quotient topology from the weak  $^*$ -topology of  $P(A)$ , the set of all pure states on  $A$ , and the topology induced from the hull-kernel topology of  $\text{Prim}(A)$ , always coincide (Dixmier [1, Theorem 3.4.11]).

**Proposition.** *The algebraic spectrum  $X$  of a  $C^*$ -algebra  $A$  is locally compact.*

**Proof.** By exactly the same argument as that in [1, Proposition 3.3.7], we can prove that for an element  $a$  in  $A$ , and a positive number  $r > 0$ , the set  $K_r$  of all  $\ker \pi$  in  $X$  such that  $\|\pi(a)\| \geq r$  is compact. We only need to change  $\pi \in \hat{A}$  to  $\ker \pi \in X$ , and note that by the same argument as in [1, Proposition 3.3.2], the mapping  $N: \ker \pi \rightarrow \|\pi(a)\|$  is lower semicontinuous on  $X$  and attains its supremum  $\|a\|$  on  $X$  since  $X$  contains the spectrum of  $A$ .

**Corollary.**  $(A, \mathcal{J})$  is locally compact.

**Proof.** Let  $[\pi] \in \hat{A}$ . Let  $U$  be a compact neighborhood of  $\ker \pi$ . Then,  $p^{-1}(U)$  is a compact neighborhood of  $[\pi]$ .

**Remark.** Kaplansky [2] has proved a similar result that the structure space  $\text{Prim}(A)$ , the set of all kernels of irreducible representations of a  $C^*$ -algebra  $A$ , is locally compact.

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