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A NOTE ON EXTREME ELEMENTS IN $A_0(K, E)$

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ABSTRACT. We give a short and simple proof to a theorem of Fakhoury, characterizing extreme elements in the unit ball of $A_0(K, E)$.

Let V be a Banach space whose dual is an L^1-space. Denote by K the unit ball of V^* equipped with the w^*-topology. Let E be a Banach space. $S(E)$ will denote its closed unit ball and ext $S(E)$ the set of extreme points in $S(E)$. $A_0(K, E)$ will be the Banach space of all the symmetric affine functions from K into E, continuous in the w^*-topology on K, and in the norm topology on E. Fakhoury has shown in [1] that if E has certain properties, then $f \in A_0(K, E)$ is extreme in $S(A_0(K, E))$ if and only if $f(\text{ext } K) \subset \text{ext } S(E)$. This result bears immediate characterization of extreme compact operators (if $E = F^*$, then $A_0(K, E)$ is precisely the space of compact operators from F into V), and generalizes similar results of Lazar [2]. By observing a simple property of spaces having the 3.2.I.P. (cf. [4] for a proper definition), and by using a selection theorem of Lazar and Lindenstrauss [3], we are able to prove Fakhoury's result in a very simple and direct way.

Lemma. *Let E be a Banach space having the* 3.2.I.P. *and let $x, y \in S(E)$, $a \in E$, such that $x \pm a \in S(E)$. Then there exists an element $b \in E$, such that $y \pm b \in S(E)$, and $\|b - a\| \leq \|y - x\|$.*

Proof. Define three closed balls in E:

$$S_1 = \{b \in E;\ \|b - y\| \leq 1\},$$

$$S_2 = \{b \in E;\ \|b + y\| \leq 1\},$$

$$S_3 = \{b \in E;\ \|b - a\| \leq \|y - x\|\}.$$

These balls intersect in pairs, for $0 \in S_1 \cap S_2$, $y + a - x \in S_1 \cap S_3$ and

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$-y + a + x \in S_2 \cap S_3$. Hence they have a nonempty intersection, and any $b \in S_1 \cap S_2 \cap S_3$ has all the desired properties.

Theorem. *Let the scalars be real, and let E be a Banach space with the* 3.2.I.P. *and let K be as above. Then an element ϕ of $A_0(K, E)$ is extreme in the unit ball of this space if and only if* $\phi(\text{ext } K) \subset \text{ext } S(E)$.

Proof. Since one direction is immediate, assume ϕ to be extreme. We make use of a selection theorem of Lazar and Lindenstrauss [3]. Define a set-valued mapping $\Sigma\colon K \to 2^{S(E)}$ by

$$\Sigma(\mu) = \{x \in S(E);\ \|\phi(\mu) \pm x\| \le 1\}, \qquad \mu \in K.$$

Now, for each $\mu \in K$, $\Sigma(\mu)$ is a norm-closed convex nonvoid subset of $S(E)$. Also $\Sigma(-\mu) = \Sigma(\mu) = -\Sigma(\mu)$ for each $\mu \in K$ (thus Σ is *symmetric*), and for each $\mu_1, \mu_2 \in K$, $0 \le \alpha \le 1$, we have

$$\alpha\Sigma(\mu_1) + (1-\alpha)\Sigma(\mu_2) \subset \Sigma(\alpha\mu_1 + (1-\alpha)\mu_2)$$

(thus Σ is *convex*). Σ is also lower semicontinuous (in the sense of Michael [5]) with the w^*-topology on K and the norm-topology on $S(E)$: Let $\mu \in K$, $x \in \Sigma(\mu)$, and $\mu_\alpha \xrightarrow{w^*} \mu$ in K. We have to show the existence of $x_\alpha \in \Sigma(\mu_\alpha)$, for each α, such that $x_\alpha \to x$ in norm. But this is immediate since $\phi(\mu_\alpha) \to \phi(\mu)$ in norm, and, for each α, the Lemma provides us with a $x_\alpha \in E$ such that $\|\phi(\mu_\alpha) \pm x_\alpha\| \le 1$ (hence $x_\alpha \in \Sigma(\mu_\alpha)$) and $\|x_\alpha - x\| \le \|\phi(\mu_\alpha) - \phi(\mu)\|$. Hence the lower semicontinuity of Σ is obvious. Now, assume that $\phi(\text{ext } K) \not\subset \text{ext } S(E)$. Hence, there is a $\mu_0 \in \text{ext } K$ and a nonzero element x_0 of $\Sigma(\mu_0)$. Now, $f(\alpha\mu_0) = \alpha x_0$, $\alpha \in [-1, 1]$, is a w^*-continuous affine symmetric selection of Σ restricted to $\text{conv}(\{\mu_0\} U - \{\mu_0\})$, for which $\{\mu_0\}$ is obviously essentially closed (cf. [3] for proper definitions). Hence, there is a w^*-continuous affine symmetric selection $\psi \in A_0(K, E)$ of Σ, such that $\psi(\mu_0) = x_0 \ne 0$. Hence, for each $\mu \in K$ one has $\|\phi(\mu) \pm \psi(\mu)\| \le 1$ and therefore $\|\phi \pm \psi\| \le 1$. Since ϕ is extreme we must have $\psi = 0$, a contradiction which completes the proof.

Remark. The estimate $\|x_\alpha - x\| \le \|\phi(\mu_\alpha) - \phi(\mu)\|$, and therefore also the requirement that E will have the 3.2.I.P., is apparently too strong for the proof of the Theorem and weaker assumptions will do as well.

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