ON THE UNIFORM ERGODIC THEOREM. II

MICHAEL LIN

ABSTRACT. Let \(|T_t|\) be a strongly continuous semigroup of bounded linear operators on a Banach space \(X\), satisfying \(\lim_{t \to \infty} \|T_t\|/t = 0\). We prove the equivalence of the following conditions: (1) \(t^{-1} \int_0^t T_s \, ds\) converges uniformly as \(t \to \infty\). (2) The infinitesimal generator \(A\) has closed range. (3) \(\lim_{\lambda \to 0^+} \lambda R_\lambda\) exists in the uniform operator topology.

In [6] the author has proved that for a bounded linear operator \(T\) on a Banach space \(X\) satisfying \(\|T^n/n\| \to 0\), \(N^{-1} \sum_{n=0}^{N-1} T^n\) converges uniformly if and only if \((I - T)X\) is closed. In this paper we obtain the analogous result for the continuous case, generalizing some results of Hille and Phillips [5, Theorem 18.8.4] without using the operational calculus devised by Hille. Abel convergence in the discrete case is treated in the appendix.

Theorem. Let \(|T_t|_{t \geq 0}\) be a strongly continuous semigroup of bounded linear operators with \(T_0 = I\), satisfying \(\|T_t/t\| \to 0\). Let \(A\) be the infinitesimal generator of \(|T_t|\) and let \(R_\lambda x = \int_0^\infty e^{-\lambda t} T_t x \, dt\) be the resolvent family \((\lambda > 0)\). Then the following conditions are equivalent:

1. There exists a bounded linear operator \(E\) such that \(\|t^{-1} \int_0^t T_s \, ds - E\|_{t \to \infty} \to 0\).
2. \(A\) has closed range.
3. \(N^{-1} \sum_{n=0}^{N-1} R^n\) converges uniformly.
4. There exists a projection \(E\) on \(\{x: T_t x = x, \forall t > 0\}\) such that \(\lim_{\lambda \to 0^+} \|\lambda R_\lambda - E\| = 0\).

Lemma 1. Under the assumptions of the theorem, \(\lim_{n \to \infty} \|(\lambda R_\lambda)^n\|/n = 0\) for every \(\lambda > 0\).

Proof. If \(t\) is large, \(\|T_t\| \leq \epsilon t\) and \(w_0 = \lim_{t \to \infty} \log \|T_t\|/t \leq \)

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lim \( t^{-1}(\log t + \log \epsilon) = 0 \). In [2, Lemma VIII.1.12] it is shown that for 
\( \lambda > 0 \geq w_0 \),
\[
R_\lambda^n x = (n - 1)!^{-1} \int_0^\infty e^{-\lambda t} t^{n-1} T_t x \, dt.
\]
By induction and integration by parts we have \((A*/w!) / 0^0 e^{-\lambda t} = 1/A\).

Fix \( \epsilon > 0 \). Then \( \|T_t\| \leq \epsilon t \) for \( t > t_0(\epsilon) \). Since \( \|T_t x\| \) is continuous on \([0, t_0]\), it is bounded, and by the principle of uniform boundedness \( \|T_t\| \leq K \) for \( 0 \leq t \leq t_0 \). Thus
\[
\frac{\|\lambda R_\lambda^n\|^n}{n!} \leq \frac{\lambda}{e^\lambda} K \int_0^{t_0} e^{-\lambda t} (t-1) dt + \frac{\lambda}{e^\lambda} \int_{t_0}^\infty e^{-\lambda t} t^{n-1} t dt \leq \frac{K \epsilon}{n + \lambda}
\]
and \( \limsup_n \|\lambda R_\lambda^n\|^n/n \leq \epsilon/e^\lambda \). Let \( \epsilon \to 0 \) to conclude the proof.

**Lemma 2.** If \( \mu R_\mu x = x \) for some \( \mu > 0 \), then \( T_t x = x \) for every \( t > 0 \).

**Proof.** By the resolvent equation \((R_\lambda - R_\mu)x = (\mu - \lambda)R_\mu R_\lambda x\).
\[
R_\lambda x - R_\mu x = R_\lambda \mu x - \lambda R_\mu R_\lambda x = R_\lambda x - \lambda R_\lambda R_\mu x.
\]
Thus
\[
(\lambda - \lambda R_\lambda)x = \mu(\lambda - \lambda R_\lambda)R_\mu x = 0,
\]
and \( \lambda R_\lambda x = x \) for every \( \lambda > 0 \). The lemma follows from the inversion formula (11.7.2) in [5].

**Remark.** If \( \sup_{t \geq 0} \|T_t\| \leq M \), we may assume \( M = 1 \). Then Lemma 1 is trivial, and Lemma 2 is proved by Falkowitz [3] without using the inversion formula.

**Proof of the Theorem.** We denote by \( R(A) \) the range of \( A \) and by \( Y \) the closure of \( R(A) \).

(2) \( \Rightarrow \) (3). For \( \lambda > 0 \) \((\lambda I - A)R_\lambda = I \) and for \( x \in D(A) \), \( R_\lambda (\lambda I - A)x = x \) [2, Lemma VIII.1.7]. Hence \( \lambda R_\lambda = I + AR_\lambda \) and \((\lambda R_\lambda - I)x \subset R(A) \). By (2) \( Y = R(A) \), so that for \( y \in Y \) there is an \( x \in D(A) \) with \( Ax = y \), and
\[
x = R_\lambda (\lambda I - A)x = \lambda R_\lambda x - \lambda R_\lambda y \quad \text{or} \quad R_\lambda y = (\lambda R_\lambda - I)x.
\]
Thus
\[
y = (\lambda I - A)R_\lambda y = (\lambda I - A)(\lambda R_\lambda - I)x = (\lambda R_\lambda - I)(\lambda I - A)x
\]
and \( Y = (\lambda R_\lambda - I)X \). Thus by applying the uniform ergodic theorem of [6] to the operator \( \lambda R_\lambda \) we obtain (3), since \( \|\lambda R_\lambda^n\|^n/n \to 0 \) by Lemma 1.

(2) \( \Rightarrow \) (1). For \( x \in D(A) \), \( AT_t x = T_t Ax \) and therefore \( T_t Y \subset Y \). The generator \( A_t \) of the restriction of \( \{T_t\} \) to \( Y \) will be the restriction of \( A \) to \( Y \cap D(A) \). It is shown above that (2) implies \( Y = (I - \lambda R_\lambda)X \), and the uniform ergodic theorem shows that \( I - \lambda R_\lambda \) is invertible on \( Y \). If \( A_t Y = 0 \)
for \( y \in Y \cap D(A) \), then
\[
y = R_\lambda(\lambda I - A)y = \lambda R_\lambda y \quad \text{or} \quad (I - \lambda R_\lambda)y = 0,
\]
and \( y = 0 \). Thus \( A_1 \) is one-to-one. On \( Y \) we have, as above, \((I - \lambda R_\lambda)Y \subset R(A_1)\). But from (2) \( \implies \) (3) we have
\[
Y \supset R(A_1) \supset (I - \lambda R_\lambda)Y = (I - \lambda R_\lambda)X = Y = R(A)
\]
and \( R(A_1) = Y \), so \( A_1^{-1} \) is defined for all of \( Y \). \( A_1 \) is closed, therefore \( A_1^{-1} \) is continuous, and by the closed graph theorem \( A_1^{-1} \) is continuous. For \( y \in Y = R(A_1) \) there is an \( x \in D(A) \cap Y \) such that \( Ax = y \) and \( \|x\| \leq \|A_1^{-1}\|y\| \). By [2, Lemma VIII.1.7]
\[
(T - I)x = \int_0^t T_\tau Ax \, d\tau = \int_0^t T_\tau y \, d\tau
\]
and
\[
\|t^{-1}\int_0^t T_\tau y \, d\tau\| \leq \|A_1^{-1}\|((\|T\| + 1)t^{-1}\|y\|). \quad \text{Hence on } Y \text{ we have uniform convergence to } 0. \quad \text{But } X = Y \oplus \{x: R_1x = x\} \text{ by the uniform ergodic theorem,}
\]
Lemma 2 then yields that \( X = Y \oplus \{x: T_\tau x = x, \forall \tau > 0\} \), and (1) follows immediately.

(3) \( \implies \) (2). By the uniform ergodic theorem \((I - R_1)X \) is closed. In (2) \( \implies \) (3) we have shown \((I - R_1)X = R(A)\) and \( R(A)\) is closed.

(1) \( \implies \) (2). It follows from (1) that \( E^2 = E \), with \( EX = \{x: T_\tau x = x, \forall \tau \geq 0\} \). As shown above, \( T_\tau Y \subset Y \). For \( x \in D(A) \), we have by [2, Lemma VIII.1.7] that \((T_\tau - I)x = \int_0^t T_\tau Ax \, d\tau \in Y \). Since \( D(A) \) is dense, we have that \( \text{clm} \bigcup_{\tau \geq 0}(I - T_\tau)X \subset Y \). On the other hand, for \( x \in D(A) \), \( Ax = \lim_{\tau \to 0}b^{-1}(T_\tau - I)x \), so that \( Y \subset \text{clm} \bigcup_{\tau \geq 0}(I - T_\tau)X \) and equality holds. Hence by (1), \( X = EX \oplus Y \) (see [2, VIII.7.2]). Thus by restricting ourselves to \( Y \) we have \( Y = \overline{R(A_1)} \) (where \( A_1 \) is the generator of the restriction, with domain \( D(A) \cap Y \)). Thus we may and do assume \( X = Y \), and \( \|t^{-1}\int_0^t T_\tau ds\| \to 0 \).

Let \( x \in D(A) \) satisfy \( Ax = 0 \). Then \( x = R_1(I - A)x = R_1x \) and by Lemma 2 \( T_\tau x = x \) for every \( \tau \geq 0 \). Since \( X = Y, x = 0 \) and \( A \) is one-to-one, with \( A^{-1} \) defined on \( R(A) \). For fixed \( t \) large enough, \( \|t^{-1}\int_0^t T_\tau ds\| < 1 \) and \( I - t^{-1}\int_0^t T_\tau ds \) is invertible, and so is \( \int_0^t (T_\tau - I) \, d\tau \). For \( y \in R(A) \) take \( x \in D(A) \) with \( Ax = y \). Then
\[
\int_0^t (T_\tau - I)x \, ds = \int_0^t \left( \int_0^s T_\tau Ax \, d\tau \right) ds = \int_0^t \left( \int_0^s T_\tau y \, d\tau \right) ds
\]
and
\[
\|x\| = \|A^{-1}y\| \leq \left\| \left( \int_0^t (T_\tau - I) ds \right)^{-1} \right\| \left\| \int_0^t \left( \int_0^s T_\tau y \, d\tau \right) ds \right\| \leq K \|y\|.
\]
Hence $A^{-1}$ is continuous with dense domain (in $X = Y$). Since $A$ is closed, $A^{-1}$ is defined on all of $Y$, and $Y = R(A) = R(A)$. Again we may and do assume $X = Y$, and $A^{-1}$ is continuous.

Let $0 < \lambda \leq \delta < 1/\|A^{-1}\|$. For $y \in Y$
\[ \|\lambda R_{\lambda}y\| = \|\lambda R_{\lambda}AA^{-1}y\| \leq \|\lambda (\lambda R_{\lambda} - I)\| \|A^{-1}\| \|y\|. \]

Hence for $0 < \lambda \leq \delta$ we have,
\[ \|\lambda R_{\lambda}\| \leq (1 + \|\lambda R_{\lambda}\|) \delta \|A^{-1}\|, \quad \text{or} \quad \|\lambda R_{\lambda}\| \leq \delta \|A^{-1}\|/\|I - \delta A^{-1}\| \equiv M. \]

Now we have
\[ \|\lambda R_{\lambda}\| \leq \|\lambda (\lambda R_{\lambda} - I)\| \|A^{-1}\| \leq \lambda (1 + M) \|A^{-1}\| \rightarrow \lambda \rightarrow 0^+ 0. \]

(4) $\Rightarrow$ (2). If $y = \lambda x$ for $x \in D(A)$, then $\lambda R_{\lambda}y = \lambda R_{\lambda}Ax = \lambda (\lambda R_{\lambda} - I)x \rightarrow 0$. Restricting ourselves to $Y$ we have $\|\lambda R_{\lambda}\|_Y \rightarrow 0$. But on $Y$ we have for $\lambda > 0$ small enough that $I - \lambda R_{\lambda}$ is invertible, hence
\[ Y = (I - \lambda R_{\lambda})Y \subset (I - \lambda R_{\lambda})X = R(A) \]
(see (2) $\Rightarrow$ (1) for last equality) and $R(A) = Y$.

Remark. [5, Theorem 18.8.4] treats more general semigroups, but is only concerned with Abel convergence, and does not include the sufficiency of $R(A)$ being closed. The method presented here is different, not using the operational calculus and spectral theory used in [5].

Corollary 1. Let $\{T_t\}$ be as above. The following two conditions are equivalent:

1. \[ t^{-1} \int_0^t T_r \, dr \text{ converges uniformly as } t \rightarrow \infty. \]

2. \[ \text{There is a } \delta > 0 \text{ such that } \sup_{0 < \lambda \leq \delta} \|R_{\lambda}y\| < \infty \text{ for every } y \in R(A). \]

If $\|T_t\| \leq M$ for $t \geq 0$, then the following (sufficient) condition is also equivalent to the previous two:

3. \[ \sup_{t \geq 0} \|\int_0^t T_r y \, dr\| < \infty \text{ for every } y \in R(A). \]

Proof. (1) $\Rightarrow$ (2). By the Theorem $R(A)$ is closed. For $y \in R(A)$ we have $x \in D(A)$ with $Ax = y$, and $R_{\lambda}y = R_{\lambda}Ax = x + \lambda R_{\lambda}x$. The proof of (1) $\Rightarrow$ (4) in the Theorem shows that $\sup_{0 < \lambda \leq \delta} \|\lambda R_{\lambda}\| < \infty$, and (2) follows. (In fact, $\sup_{0 < \lambda \leq \delta} \|\lambda R_{\lambda}\| < \infty$ by the computations of Lemma 1, $\sup_{\delta \leq \lambda} \|\lambda R_{\lambda}\| < \infty$.)
(2) \implies (1). We look at the semigroup restricted to \( Y = \overline{R(A)} \), with generator \( A_1 \).

By the principle of uniform boundedness, \( \sup_{0 < t < s} \| R_t \|_Y < \infty \) and therefore \( \lim_{\lambda \to 0} \| \lambda R_\lambda \|_Y = 0 \). By the Theorem, \( Y = R(A_1) \subset R(A) \). Hence \( R(A) \) is closed and (1) follows from the Theorem.

(3) \implies (1) is proved similarly.

If \( \| T_t \| \leq M \) for every \( t \geq 0 \) and we assume (1), then for \( y = Ax \) \((x \in D(A))\) we have, by [2, Lemma VIII.1.7],

\[
\left\| \int_0^t T_r y \, dr \right\| = \left\| \int_0^t T_r Ax \, dr \right\| = \left\| (T_t - I)x \right\| \leq (M + 1) \| x \|.
\]

Since \( R(A) \) is closed, (3) follows.

**Corollary 2.** Assume \( \| T_t \| \leq M \) for every \( t \geq 0 \). Let \( \nu \) be a probability measure on \([0, \infty)\), \( \nu([0]) < 1 \), and define \( Ux = \int T_t x \, d\nu \). If \( N^{-1} \sum_{n=0}^{N-1} U^n \) converges uniformly, then \( t^{-1} \int_0^t T_r \, dr \) converges uniformly \((\text{as } t \to \infty)\).

**Proof.** Since \( \| x \| = \sup_{t \geq 0} \| T_t x \| \) is an equivalent norm for \( X \) such that \( \| T_t \| \leq 1 \), we may and do assume \( \| T_t \| \leq 1 \).

Note that \( U \) is well defined in the strong operator topology, by [5, Theorem 3.7.4].

By the discrete uniform ergodic theorem \( X = (I - U)X \oplus \{ x: Ux = x \} \); since \( T_t \) commutes with \( U \), both subspaces are invariant under \( \{ T_t \} \), and we can restrict ourselves to each one. Hence it suffices to treat two cases:

1. \( I - U \) is invertible.
2. \( U = I \).

**Case 1.** Fix \( \epsilon > 0 \). Let \( \beta > 0 \) be such that \( \nu([0, \beta]) > 1 - \epsilon \). Define \( \nu_\epsilon(C) = \nu(C \cap [0, \beta]) / \nu([0, \beta]) \). Then \( \| \nu_\epsilon - \nu \| \leq 1/(1 - \epsilon) - 1 + \epsilon \). Define \( U_\epsilon x = \int T_t x \, d\nu_\epsilon \). Then

\[
\left\| (I - U) - (I - U_\epsilon) \right\| = \left\| U - U_\epsilon \right\| \leq \| \nu - \nu_\epsilon \| \to 0 \quad \text{as } \epsilon \to 0
\]

and \( I - U_\epsilon \) is also invertible for \( \epsilon \) small enough. Thus we may and do assume that \( \nu \) is supported on a finite interval \([0, \beta]\).

Let \( A \) be the infinitesimal generator of \( \{ T_t \} \). Since \( D(A) \) is dense, for \( y \in X \) there is a sequence \( x_n \in D(A) \) with \( x_n \to (I - U)^{-1} y \). Hence

\[
y = \lim_{n \to \infty} (I - U)x_n = \lim_{n \to \infty} \int (I - T_t)x_n \, d\nu = -\lim_{n \to \infty} \int \left( \int_0^t T_r Ax_n \, dr \right) \, d\nu,
\]

which shows that \( y \in \overline{R(A)} \) \((\text{since } \overline{R(A)} \text{ is invariant under } \{ T_t \})\). Let \( z_n \in D(A) \) satisfy \( Az_n \to y \). Then
\[
\|(I - U)z_n - (I - U)z_m\| \leq \int_0^t \left\{ \int_0^r \|Az_n - Az_m\| \, dr \right\} \, dv
\]
\[
= \int_0^t \|Az_n - Az_m\| \, dv \leq \beta \|Az_n - Az_m\|.
\]
Thus \(\{(I - U)z_n\}\) is a Cauchy sequence, and \(z_n = (I - U)^{-1}(I - U)z_n\) converges (strongly), say to \(z\). Then \(z_n \to z\) and \(Az_n \to y\), and since \(A\) is a closed operator, \(z \in D(A)\) and \(y = Az\). Thus \(X = R(A)\) and by the theorem
\[
\lim_{t \to 0} t^{-1} \int_0^t T_r \, dr = 0.
\]

**Case 2.** We show first that \(U = I\) implies \(T_{t_0} = I\) for some \(t_0 > 0\). If, for \(t_0 > 0\), \(\alpha = \nu(t_0) > 0\), then for every \(x \in X\) we have \(x = Ux = \alpha T_{t_0} x + (1 - \alpha) \int T_r x \, dv_1\), where \(\nu_1 = (\nu - \alpha \delta_{t_0})/(1 - \alpha)\) with \(\delta_{t_0}\) the Dirac measure at \(t_0\). By [3, Lemma 1], \(T_{t_0} x = x\), and thus \(T_{t_0} = I\). Assume now that \(\nu(\{t\}) = 0\) for \(t > 0\). Let \(F(t) = \nu((- \infty, t])\) be the distribution of \(\nu\). Then \(F(t)\) is continuous at \(t > 0\), and since \(F(0) = \nu(0) < 1\), there is a \(t' > 0\) with \(0 < F(t') = a < 1\). Let \(t_0 = \sup \{t: F(t) = a\}\), and \(F(t_0) = a\) by continuity. Take \(t_0 < t_n \leq t_1\), and let \(a_n = \nu(\{t_0, t_n\})\). Define (since \(1 > a_n > 0\))
\[
\nu_n(C) = \nu(C \cap [t_0, t_n])/a_n \quad \text{and} \quad \mu_n(C) = \{\nu(C) - a_n \nu_n(C)\}/(1 - a_n).
\]
Then \(\nu_n\) and \(\mu_n\) are nonzero probability measures with \(a_n \nu_n + (1 - a_n) \mu_n = \nu\). By [3, Lemma 1] we also have that \(\int T_r x \, dv_n = x\) for every \(x \in X\), so that \(\int T_r \, dv_n = I\). Fix \(x \in X\), \(x^* \in X^*\). Then \(\langle x^*, T_t x \rangle\) is continuous on \([t_0, t_n]\) and there is a point \(t_0 \leq s_n \leq t_n\) such that \(\langle x^*, T_s x \rangle = \int \langle x^*, T_r x \rangle \, dv_n = \langle x^*, x \rangle\). Since also \(\lim s_n = t_0\), the continuity of \(\{T_t\}\) yields \(\langle x^*, T_{t_0} x \rangle = \langle x^*, x \rangle\). Hence \(T_{t_0} = I\). To finish the proof in Case 2, denote \(n = \lfloor t/t_0 \rfloor\). Then for \(t > t_0\) we have (with \(\alpha = t/t_0 - n\))
\[
t^{-1} \int_0^t T_r \, dr = t^{-1} \int_0^{nt_0} T_r \, dr + t^{-1} \int_{nt_0}^t T_r \, dr
\]
\[
= t^{-1} \int_0^{nt_0} T_r \, dr + t^{-1} \int_{nt_0}^t T_r \, dr \to t^{-1} \int_0^{nt_0} T_r \, dr \quad \text{as} \quad t \to t_0^+.
\]

**Remarks.** (1) The corollary can be applied if \(I - U - Q\) is invertible for some compact operator \(Q\) (see [6]). (2) In the discrete case \(\{T^n\}\) treated in [6] there is a requirement \(\sum_{i=1}^k a_i = 1\). It can be removed by an approxima-
tion argument, as in the proof of Corollary 2. (Also see the example in the appendix.)

Appendix. We show here that if \( \|T^n/n\| \to 0 \), then Abel and Cesàro convergence of \( \{T^n\} \) are equivalent in the uniform operator topology. Hille [4] proved it for the case \( \sup\|T^n\| < \infty \), using a general method which does not apply in the case that \( T \) is not power-bounded. (An extension of Hille's result in the strong operator topology is a consequence of [1, Theorem 1].)

**Proposition.** Let \( T \) be a linear operator on \( X \) with \( \|T^n/n\| \to 0 \). Then the following conditions are equivalent:

1. \( N^{-1} \sum_{n=0}^{N-1} T^n \) converges in the uniform operator topology.
2. \( \lim_{r \to 1^-} \|(1 - r) \sum_{n=0}^{\infty} r^n T^n - E\| = 0 \) for some operator \( E \).
3. \( (I - T)X \) is closed.

**Proof.** Let \( Y = (I - T)X \), which is invariant under \( T \), and let \( S \) on \( Y \) be the restriction of \( T \).

(1) \( \Rightarrow \) (2). Since this follows from Hille's general results [4], we only sketch a simple proof for the present situation. From (1) we have \( X = Y \oplus \{x: Tx = x\} \), and we have only to show \( \|(1 - r) \sum_{n=0}^{\infty} r^n S^n\| \to 0 \). But by the uniform ergodic theorem, \( I - S \) is invertible on \( Y \). Thus, for \( 0 < r < 1 \),

\[
(1 - r) \sum_{n=0}^{\infty} r^n S^n = (1 - r) \sum_{n=0}^{\infty} r^n S^n (I - S) (I - S)^{-1}
\]

\[
\leq \left\{ \left\| (1 - r) \sum_{n=1}^{\infty} (r^n - r^{n-1}) S^n \right\| + (1 - r) \right\} \left\| (I - S)^{-1} \right\|
\]

\[
\leq (1 - r) \left\| (I - S)^{-1} \right\| \left\{ 1 + (1 - r) \sum_{n=1}^{\infty} r^{n-1} \|S^n\| \right\}.
\]

Fix \( \epsilon > 0 \). For \( n > k \), \( \|S^n\|/n < \epsilon \) and

\[
(1 - r)^2 \sum_{n=1}^{\infty} r^{n-1} \|S^n\| \leq (1 - r)^2 \sum_{n=1}^{k} r^{n-1} \|S^n\| + \epsilon (1 - r)^2 \sum_{n=1}^{\infty} r^{n-1} n
\]

\[
= (1 - r)^2 \sum_{n=1}^{k} r^{n-1} \|S^n\| + \epsilon \to \epsilon.
\]

Let \( \epsilon \to 0 \) to conclude (2).

(2) \( \Rightarrow \) (3). We shall show \( (I - S)Y = Y \), which implies (3). Hence we restrict ourselves to \( Y \). If \( y = (I - T)x \), then
\[(1 - r) \sum_{n=0}^{\infty} r^n S^ny = (1 - r) \sum_{n=0}^{\infty} r^n(I - T)T^nx\]

\[= (1 - r) \sum_{n=0}^{\infty} r^nT^nx - (1 - r)r^{-1} \sum_{n=0}^{\infty} r^nT^nx + (1 - r)r^{-1}x\]

\[\rightarrow Ex - Ex + 0 = 0.\]

Hence \(\|(1 - r) \sum_{n=0}^{\infty} r^nS^n\| \rightarrow r - 1^{-}.\) Thus for fixed \(0 < r < 1,\) close enough to 1, \(I - (1 - r) \sum_{n=0}^{\infty} r^nS^n\) is invertible. Since \(\sum_{n=0}^{\infty} r^nS^n\) is defined in the uniform operator topology (by virtue of \(\|T^n\|/n \rightarrow 0\)), we have that

\[(1 - r) \sum_{n=1}^{\infty} r^n \left( \sum_{i=0}^{n-1} S^i \right) = (1 - r) \sum_{i=0}^{\infty} \left( \sum_{n=i+1}^{\infty} r^n \right) S^i = \sum_{i=0}^{\infty} r^{i+1}S^i\]

is also defined in the uniform operator topology. But

\[(I - S)\sum_{n=0}^{\infty} r^nS^n = (1 - r) \sum_{n=0}^{\infty} r^n(I - S^n) = \sum_{n=1}^{\infty} r^n \left( \sum_{i=0}^{n-1} S^i \right)\]

and \((I - S)Y = Y\) by invertibility of the left-hand side.

(3) \(\Rightarrow\) (1) is proved in [6].

Example. We exhibit an operator \(T\) satisfying \(\|T\| < 1,\) \((I - T)X = X\) and \((I - T^2)X\) is not closed. Since \(\|N^{-1} \sum_{n=0}^{N-1} T^n\| \rightarrow 0,\) \(T\) satisfies the conditions of [6, Corollary 2] (with \(Q = 0\)), but is not quasi-compact (since this implies \(T^k\) are all quasi-compact).

Let \(X = l_2,\) \(0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > -1.\) Define \(Tx = \{x_i, i\}\) where \(x = \{x_i\}.\) Then \((I - T)^{-1} x = \{x_i/(\lambda_i - 1)^{-1}\},\) \((I - T^2)X\) contains all sequences with finitely many nonzero terms, and is dense. But \(-1 \in \sigma(T)\) and \(I + T\) is not invertible, but \(I + T\) is one-to-one. Hence, \((I - T^2)X \subseteq (I + T)X \neq X = (I - T^2)X.\)

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210