A CLASS OF MANIFOLDS COVERED
BY EUCLIDEAN SPACE

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ABSTRACT. The following is the main result:

Theorem 1. Suppose \( W^n \) is a PL manifold which has homotopy type \( K(\Gamma, 1) \), \( W \) has one end \( \infty \), \( \pi_1 \) is essentially constant at \( \infty \), and the induced homomorphism \( \pi_1(\infty) \rightarrow \pi_1(W) \) is an isomorphism. Then the universal cover of \( W \) is PL homomorphic to \( \mathbb{R}^n \) provided \( n \geq 5 \).

1. Introduction. Let \( W^n \) denote a topological manifold of dimension \( n \). \( W \) is said to have homotopy type \( K(\Gamma, 1) \) provided \( \pi_1(W) \cong \Gamma \) and the universal cover \( \tilde{W} \) of \( W \) is contractible. A natural question is: Under what conditions is \( W \) homeomorphic to \( \mathbb{R}^n \)? For \( n = 1, 2 \), no additional assumptions are necessary since \( W \) contractible implies it is homeomorphic to \( \mathbb{R}^n \). For \( n \geq 3 \), the existence of contractible manifolds \( W \) not homeomorphic to \( \mathbb{R}^n \) makes some restrictions likely. In the next section an example is given of a 3-dimensional, open (noncompact and empty boundary) manifold of homotopy type \( K(\Gamma, 1) \) whose universal cover is not homeomorphic to \( \mathbb{R}^3 \). It is a long-standing conjecture that the universal cover of a closed (compact and empty boundary) manifold \( W^n \) of homotopy type \( K(\Gamma, 1) \), \( \Gamma \) an infinite group, is homeomorphic to \( \mathbb{R}^n \). Some partial results to this conjecture in the case \( n \geq 5 \) can be found (without proof) in [3]. In [7] Waldhausen gives sufficient conditions for the universal cover of a closed 3-manifold of homotopy type \( K(\Gamma, 1) \) to be homeomorphic to \( \mathbb{R}^3 \). The author knows of no counterexample to this conjecture.

This paper is concerned with open \( K(\Gamma, 1) \) manifolds which admit a piecewise linear structure. For the remainder of this paper let \( W^n \) denote such a manifold, and suppose further that \( W \) has a single end which is denoted by \( \infty \). An inverse sequence of groups

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is said to be essentially constant if there exists a subsequence

so that isomorphisms \( \text{Im}(f_1') \cong \text{Im}(f_2') \cong \text{Im}(f_3') \) are induced. Following [5], we say that \( \pi_1 \) is essentially constant at \( \infty \) if for a sequence \( K_1 \subset K_2 \subset K_3 \) of compacta with \( W = \bigcup \text{Int} K_i \), the sequence induced by inclusion

is essentially constant. The choice of \( \{K_i \} \) of base points, and of connecting base paths used to define \( \widetilde{\delta} \) does not affect this property. Also \( \pi_1(\infty) = \text{proj lim} (\widetilde{\delta}) \) proves to be independent of these choices up to isomorphism in a preferred conjugacy class; and \( \pi_1(W - K_i) \rightarrow \pi_1(W) \), for large \( i \), induces a homomorphism \( \pi_1(\infty) \rightarrow \pi_1(W) \) which is again determined within a conjugacy class. For a verification of these facts see [6].

Let \( M \) denote a compact manifold with nonempty boundary of homotopy type \( K(\Pi, 1) \). If \( \pi_1(\partial M) \) is isomorphic by inclusion to \( \pi_1(M) \), then \( W = \text{Int} M \) is an open manifold with one end \( \infty \), \( \pi_1 \) is essentially constant at \( \infty \), and \( \pi_1(\infty) \rightarrow \pi_1(W) \) is an isomorphism. The above properties of \( W \) are, however, not sufficient to guarantee that \( W \) is the interior of a compact manifold. (See [5].)

The following is the main result.

**Theorem 1.** Suppose \( W^n \) is a PL manifold which has homotopy type \( K(\Pi, 1) \), \( W \) has one end \( \infty \), \( \pi_1 \) is essentially constant at \( \infty \), and the induced homomorphism \( \pi_1(\infty) \rightarrow \pi_1(W) \) is an isomorphism. Then \( W \) is PL homeomorphic to \( R^n \) provided \( n \geq 5 \).

Before proceeding with the proof of Theorem 1, we first make an application of Theorem 1. Let \( \Pi \) denote a countable group with finite cohomological dimension, denoted \( \text{cd}(\Pi) \). In [3] it is shown that if \( \Pi \) is countable and \( \text{cd}(\Pi) \) is finite, then there is a polyhedron \( X^n \) of type \( K(\Pi, 1) \) of dimension \( n = \text{cd}(\Pi) \) if \( \text{cd}(\Pi) \neq 2 \), and of dimension \( n = 3 \) if \( \text{cd}(\Pi) = 2 \). Next it is shown that \( X^n \) is homotopy equivalent to a subpolyhedron \( Y^n \) of \( R^{2n} \). Let \( W \) denote the interior of a regular neighborhood of \( Y \). When \( \text{cd}(\Pi) \geq 2 \), \( Y \) has co-dimension \( \geq 3 \) in \( W \) and therefore \( W \) has one end \( \infty \), \( \pi_1 \) is essentially con-
stant at $\infty$, and $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. We have the following generalization of Corollary 3 of [3].

**Theorem 2.** Let $\Pi$ be a countable group of finite cohomological dimension $n$. If $n \neq 2$, there is a covering action of $\Pi$ on $R^{2n}$, and if $n = 2$ there is a covering action of $\Pi$ on $R^6$.

**Proof.** If $n \geq 2$ the theorem follows from preceding remarks together with Theorem 1. If $n = 1$, this is Theorem 2 of [3].

All work is done in the PL category. [2] and [8] form standard references. Int and $\partial$ are used to denote interior and boundary, respectively, a superscript is used to denote dimension and "$\simeq$" denotes "is PL homeomorphic to".

2. First we prove two lemmas.

**Lemma 1.** Suppose $W^n$, $n \geq 5$, is a PL manifold which has homotopy type $K(\Pi, 1)$, $W$ has a single end $\infty$, $\pi_1$ is essentially constant at $\infty$, and the induced homomorphism $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. Then given a compact set $C$ in $W$, there is a compact set $D$ containing $C$ such that $\pi_i(W, W - D) = 0$ for $i = 0, 1, 2$.

**Proof.** From [5, Proposition 1.9, Part A], we have the existence of an arbitrarily small connected neighborhood $V$ of $\infty$, such that in the following commutative diagram

$$
\begin{array}{ccc}
\pi_1(\infty) & \xrightarrow{j_*} & \pi_1(V) \\
\downarrow{i_*} & & \downarrow{i_*} \\
\pi_1(W) & & \\
\end{array}
$$

$j_*$ is an isomorphism by construction, $i_*$ is assumed to be an isomorphism, hence $l_*$ is an isomorphism. But $\pi_2(W) = 0$ and from the homotopy sequence of the pair $(W, V)$, we have $\pi_i(W, V) = 0$ for $i = 0, 1, 2$. By the definition of neighborhood of $\infty$, $W - V$ is compact. Therefore given a compact set $C$, let $V$ be a connected neighborhood of $\infty$ such that $V \cap C = \emptyset$, and such that $\pi_1(\infty) \to \pi_1(V)$ is an isomorphism. Then $W - V$ is compact, contains $C$, and $\pi_i(W, V) = 0$ for $i = 0, 1, 2$.

**Lemma 2.** Let $K$ be a triangulation of the manifold $W$. Let $J$ denote
the i-skeleton of \( K \), and \( L \) the dual \((n - i - 1)\)-skeleton of \( K \). Let \( C \) denote a compact subset of \( W \) such that \( C \cap L = \emptyset \). Then there exists a finite subcomplex \( J_0 \) of \( J \) and a homeomorphism \( k \) of \( W \) such that

\[
\begin{align*}
(1) & \quad k|_{J \cup L} = 1, \\
(2) & \quad k(U) \supset C,
\end{align*}
\]

where \( U \) denotes the simplicial neighborhood of \( J_0 \) in \( K^n \).

Proof. As usual let \( N(J, K^n) \) and \( N(L, K^n) \) denote simplicial neighborhoods of \( J \) and \( L \), respectively, in \( K^n \). Observe that since \( L \) is the dual skeleton to \( J \), \( d(N(J, K^n)) = d(N(L, K^n)) \), and from the mapping cylinder structure of derived neighborhoods,

\[
N(L, K^n) - L \cong \partial(N(L, K^n)) \times [0, 1).
\]

Let

\[
Y = N(J, K^n) \cup (N(L, K^n) - L).
\]

But \( C \cap L = \emptyset \), so \( C \subseteq Y \), and since \( C \) is compact, one can use the product structure on \( N(L, K^n) - L \) to find a homeomorphism \( k: Y \to Y \) with compact support such that \( k(N(J, K^n)) \supset C \) and \( k|_J = 1 \). Extend \( k \) by the identity to \( L \). Let

\[
J_0 = \{ \sigma \in J : k^{-1}(C) \cap N(\sigma, K^n) \neq \emptyset \}.
\]

The compactness of \( C \) implies \( J_0 \) is finite and from the definition of \( J_0 \),

\[
k(N(J_0, K^n)) \supset C.
\]

Proof of Theorem 1. From [1] it suffices to show that given a compact set \( D \) contained in \( \tilde{W} \), \( D \) is contained in the interior of an \( n \)-ball.

Let \( K \) denote a triangulation of \( W \), \( \tilde{K} \) the induced triangulation of \( \tilde{W} \). Let \( J \) denote the \((n - 3)\)-skeleton of \( K \), and \( L \) the dual 2-skeleton. Let \( \tilde{p} \) denote the projection of \( W \) onto \( W \) and let \( C = \tilde{p}(D) \). \( C \) is compact. By Lemma 1, there is a neighborhood \( V \) of \( \infty \) such that

\[
(1) \quad C \subset W - V,
\]

\[
(2) \quad \pi_i(W, V) = 0, \quad i = 0, 1, 2.
\]

From standard engulfing, as in [2], there is a homeomorphism \( h: W \to W \) with compact support such that \( h(V) \) contains \( L \). Thus \( h(C) \cap L = \emptyset \). Let \( \tilde{h} \) denote the homeomorphism on \( \tilde{W} \) which covers \( h \). Then \( \tilde{h}(D) \cap \tilde{L} = \emptyset \). Let \( D' = \tilde{h}(D) \). From Lemma 2, one can find a finite subcomplex \( \tilde{J}_0 \) of \( \tilde{J} \) and a homeomorphism \( k: \tilde{W} \to \tilde{W} \) such that \( k(N(\tilde{J}_0, K^n)) \supset D' \). Now \( \tilde{W} \) is contractible, \( \dim J_0 \leq n - 3 \), hence \( J_0 \) lies in the interior of a ball. But \( N(\tilde{J}_0, K^n) \) collapses to \( J_0 \), hence \( N(J_0, K^n) \) lies in the interior of a ball.

Let \( B' \) denote such a ball. Then \( k(B) \supset D' \) and \( \tilde{h}(B) = k(B) \cap D \).
The following is an outline of the construction of the 3-dimensional example promised in the introduction. Let $M^3$ denote an open $K(\Pi, 1)$ manifold, and let $W^3$ denote an open, contractible 3-manifold which does not embed in $R^3$ [4]. Let $L_1$ and $L_2$ denote the images of $[0, \infty)$ under PL embeddings in $M$ and $W$, respectively. Let $M^\prime$ and $W^\prime$ denote the complements in $M$ and $W$, respectively, of the interior of a regular neighborhood of each of $L_1$ and $L_2$. Then $\partial M^\prime \cong \partial W^\prime \cong R^2$. Let $N = M^\prime \cup_h W^\prime$, where $h$ is an orientation preserving homeomorphism from $\partial M^\prime$ onto $\partial W^\prime$. Then the universal cover $\tilde{N}$ of $N$ is an open contractible manifold and so $\tilde{N}$ is a $K(\Pi, 1)$ manifold. Also $W^\prime \subset \tilde{N}$ lifts to infinitely many disjoint copies of $W^\prime$ in $\tilde{N}$. But $\text{int} \ W^\prime$ is homeomorphic to $W$ and so $\tilde{N}$ cannot be homeomorphic to $R^3$. 

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