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A CLASS OF MANIFOLDS COVERED BY EUCLIDEAN SPACE

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ABSTRACT. The following is the main result:

Theorem 1. *Suppose W^n is a PL manifold which has homotopy type $K(\Pi, 1)$, W has one end ∞, π_1 is essentially constant at ∞, and the induced homomorphism $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. Then the universal cover of W is PL homomorphic to R^n provided $n \geq 5$.*

1. **Introduction.** Let W^n denote a topological manifold of dimension n. W is said to have homotopy type $K(\Pi, 1)$ provided $\pi_1(W) \simeq \Pi$ and the universal cover $\tilde{W}$ of W is contractible. A natural question is: Under what conditions is $\tilde{W}$ homeomorphic to R^n? For $n = 1, 2$, no additional assumptions are necessary since $\tilde{W}$ contractible implies it is homeomorphic to R^n. For $n \geq 3$, the existence of contractible manifolds W not homeomorphic to R^n makes some restrictions likely. In the next section an example is given of a 3-dimensional, open (noncompact and empty boundary) manifold of homotopy type $K(\Pi, 1)$ whose universal cover is not homeomorphic to R^3. It is a long-standing conjecture that the universal cover of a closed (compact and empty boundary) manifold W^n of homotopy type $K(\Pi, 1)$, Π an infinite group, is homeomorphic to R^n. Some partial results to this conjecture in the case $n \geq 5$ can be found (without proof) in [3]. In [7] Waldhausen gives sufficient conditions for the universal cover of a closed 3-manifold of homotopy type $K(\Pi, 1)$ to be homeomorphic to R^3. The author knows of no counterexample to this conjecture.

This paper is concerned with open $K(\Pi, 1)$ manifolds which admit a piecewise linear structure. For the remainder of this paper let W^n denote such a manifold, and suppose further that W has a single end which is denoted by ∞. An inverse sequence of groups

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$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \xleftarrow{f_3} \cdots$$

is said to be essentially constant if there exists a subsequence

$$G_{i_1} \xleftarrow{f'_1} G_{i_2} \xleftarrow{f'_2} G_{i_3} \xleftarrow{f'_3} \cdots$$

so that isomorphisms $\mathrm{Im}(f'_1) \cong \mathrm{Im}(f'_2) \cong \mathrm{Im}(f'_3)$ are induced. Following [5], we say that π_1 is essentially constant at ∞ if for a sequence $K_1 \subset K_2 \subset K_3$ of compacta with $W = \bigcup \mathrm{int}\, K_i$, the sequence induced by inclusion

$$\mathcal{S}: \pi_1(W - K_1) \leftarrow \pi_1(W - K_2) \leftarrow \pi_1(W - K_3) \leftarrow \cdots$$

is essentially constant. The choice of $\{K_i\}$, of base points, and of connecting base paths used to define $\mathcal{S}$ does not affect this property. Also $\pi_1(\infty) = \mathrm{proj\ lim}(\mathcal{S})$ proves to be independent of these choices up to isomorphism in a preferred conjugacy class; and $\pi_1(W - K_i) \to \pi_1(W)$, for large i, induces a homomorphism $\pi_1(\infty) \to \pi_1(W)$ which is again determined within a conjugacy class. For a verification of these facts see [6].

Let M denote a compact manifold with nonempty boundary of homotopy type $K(\Pi, 1)$. If $\pi_1(\partial M)$ is isomorphic by inclusion to $\pi_1(M)$, then $W = \mathrm{int}\, M$ is an open manifold with one end ∞, π_1 is essentially constant at ∞, and $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. The above properties of W are, however, not sufficient to guarantee that W is the interior of a compact manifold. (See [5].)

The following is the main result.

Theorem 1. *Suppose W^n is a PL manifold which has homotopy type $K(\Pi, 1)$, W has one end ∞, π_1 is essentially constant at ∞, and the induced homomorphism $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. Then $\tilde{W}$ is PL homeomorphic to R^n provided $n \geq 5$.*

Before proceeding with the proof of Theorem 1, we first make an application of Theorem 1. Let Π denote a countable group with finite cohomological dimension, denoted $\mathrm{cd}(\Pi)$. In [3] it is shown that if Π is countable and $\mathrm{cd}(\Pi)$ is finite, then there is a polyhedron X^n of type $K(\Pi, 1)$ of dimension $n = \mathrm{cd}(\Pi)$ if $\mathrm{cd}(\Pi) \neq 2$, and of dimension $n = 3$ if $\mathrm{cd}(\Pi) = 2$. Next it is shown that X^n is homotopy equivalent to a subpolyhedron Y^n of R^{2n}. Let W denote the interior of a regular neighborhood of Y. When $\mathrm{cd}(\Pi) \geq 2$, Y has codimension ≥ 3 in W and therefore W has one end ∞, π_1 is essentially con-

stant at ∞, and $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. We have the following generalization of Corollary 3 of [3].

Theorem 2. *Let Π be a countable group of finite cohomological dimension n. If $n \neq 2$, there is a covering action of Π on R^{2n}, and if $n = 2$ there is a covering action of Π on R^6.*

Proof. If $n \geq 2$ the theorem follows from preceding remarks together with Theorem 1. If $n = 1$, this is Theorem 2 of [3].

All work is done in the PL category. [2] and [8] form standard references. Int and ∂ are used to denote interior and boundary, respectively, a superscript is used to denote dimension and "$\simeq$" denotes "is PL homeomorphic to".

2. First we prove two lemmas.

Lemma 1. *Suppose W^n, $n \geq 5$, is a PL manifold which has homotopy type $K(\Pi, 1)$, W has a single end ∞, π_1 is essentially constant at ∞, and the induced homomorphism $\pi_1(\infty) \to \pi_1(W)$ is an isomorphism. Then given a compact set C in W, there is a compact set D containing C such that $\pi_i(W, W - D) = 0$ for $i = 0, 1, 2$.*

Proof. From [5, Proposition 1.9, Part A], we have the existence of an arbitrarily small connected neighborhood V of ∞, such that in the following commutative diagram

$$\begin{tikzcd} \pi_1(\infty) \arrow[rr, "j_*"] \arrow[dr, "i_*"'] & & \pi_1(V) \\ & \pi_1(W) \arrow[ur, "l_*"'] & \end{tikzcd}$$

j_* is an isomorphism by construction, i_* is assumed to be an isomorphism, hence l_* is an isomorphism. But $\pi_2(W) = 0$ and from the homotopy sequence of the pair (W, V), we have $\pi_i(W, V) = 0$ for $i = 0, 1, 2$. By the definition of neighborhood of ∞, $W - V$ is compact. Therefore given a compact set C, let V be a connected neighborhood of ∞ such that $V \cap C = \emptyset$, and such that $\pi_1(\infty) \to \pi_1(V)$ is an isomorphism. Then $W - V$ is compact, contains C, and $\pi_i(W, V) = 0$ for $i = 0, 1, 2$.

Lemma 2. *Let K be a triangulation of the manifold W. Let J denote*

the i-skeleton of K, and L the dual $(n-i-1)$-skeleton of K. Let C denote a compact subset of W such that $C \cap L = \emptyset$. Then there exists a finite subcomplex J_0 of J and a homeomorphism k of W such that

(1) $k|_{J \cup L} = 1$,

(2) $k(U) \supset C$,

where U denotes the simplicial neighborhood of J_0 in K''.

Proof. As usual let $N(J, K'')$ and $N(L, K'')$ denote simplicial neighborhoods of J and L, respectively, in K''. Observe that since L is the dual skeleton to J, $\partial(N(J, K'')) = \partial(N(L, K''))$, and from the mapping cyclinder structure of derived neighborhoods,

$$N(L, K'') - L \simeq \partial(N(L, K'')) \times [0, 1).$$

Let

$$Y = N(J, K'') \cup (N(L, K'') - L).$$

But $C \cap L = \emptyset$, so $C \subset Y$, and since C is compact, one can use the product structure on $N(L, K'') - L$ to find a homeomorphism $k: Y \to Y$ with compact support such that $k(N(J, K'')) \supset C$ and $k|J = 1$. Extend k by the identity to L. Let

$$J_0 = \{\sigma \in J: k^{-1}(C) \cap N(\sigma, K'') \neq \emptyset\}.$$

The compactness of C implies J_0 is finite and from the definition of J_0, $k(N(J_0, K'')) \supset C$.

Proof of Theorem 1. From [1] it suffices to show that given a compact set D contained in $\tilde{W}$, D is contained in the interior of an n-ball.

Let K denote a triangulation of W, $\tilde{K}$ the induced triangulation of $\tilde{W}$. Let J denote the $(n-3)$-skeleton of K, and L the dual 2-skeleton. Let p denote the projection of $\tilde{W}$ onto W and let $C = p(D)$. C is compact. By Lemma 1, there is a neighborhood V of ∞ such that

(1) $C \subset W - V$,

(2) $\pi_i(W, V) = 0$, $i = 0, 1, 2$.

From standard engulfing, as in [2], there is a homeomorphism $h: W \to W$ with compact support such that $h(V)$ contains L. Thus $h(C) \cap L = \emptyset$. Let $\tilde{h}$ denote the homeomorphism on $\tilde{W}$ which covers h. Then $\tilde{h}(D) \cap \tilde{L} = \emptyset$. Let $D' = \tilde{h}(D)$. From Lemma 2, one can find a finite subcomplex $\tilde{J}_0$ of $\tilde{J}$ and a homeomorphism $k: \tilde{W} \to \tilde{W}$ such that $k(N(\tilde{J}_0, \tilde{K}'')) \supset D'$. Now $\tilde{W}$ is contractible, $\dim \tilde{J}_0 \leq n-3$, hence $\tilde{J}_0$ lies in the interior of a ball. But $N(\tilde{J}_0, \tilde{K}'')$ collapses to $\tilde{J}_0$, hence $N(\tilde{J}_0, \tilde{K}'')$ lies in the interior of a ball. Let B denote such a ball. Then $k(B) \supset D'$ and $\tilde{h}^{-1}k(B) \supset D$.

The following is an outline of the construction of the 3-dimensional example promised in the introduction. Let M^3 denote an open $K(\Pi, 1)$ manifold, and let W^3 denote an open, contractible 3-manifold which does not embed in R^3 [4]. Let L_1 and L_2 denote the images of $[0, \infty)$ under PL embeddings in M and W, respectively. Let M' and W' denote the complements in M and W, respectively, of the interior of a regular neighborhood of each of L_1 and L_2. Then $\partial M' \simeq \partial W' \simeq R^2$. Let $N = M' \cup_h W'$, where h is an orientation preserving homeomorphism from $\partial M'$ onto $\partial W'$. Then the universal cover $\tilde{N}$ of N is an open contractible manifold and so N is a $K(\Pi, 1)$ manifold. Also $W' \subset N$ lifts to infinitely many disjoint copies of W' in $\tilde{N}$. But int W' is homeomorphic to W and so $\tilde{N}$ cannot be homeomorphic to R^3.

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