A CLASS OF MANIFOLDS COVERED BY EUCLIDEAN SPACE

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ABSTRACT. The following is the main result:

Theorem 1. Suppose \( W^n \) is a PL manifold which has homotopy type \( K(\Pi, 1) \), \( W \) has one end \( \infty \), \( \pi_1 \) is essentially constant at \( \infty \), and the induced homomorphism \( \pi_1(\infty) \to \pi_1(W) \) is an isomorphism. Then the universal cover of \( W \) is PL homomorphic to \( \mathbb{R}^n \) provided \( n \geq 5 \).

1. Introduction. Let \( W^n \) denote a topological manifold of dimension \( n \). \( W \) is said to have homotopy type \( K(\Pi, 1) \) provided \( \pi_1(W) \cong \Pi \) and the universal cover \( \tilde{W} \) of \( W \) is contractible. A natural question is: Under what conditions is \( W \) homeomorphic to \( \mathbb{R}^n \)? For \( n = 1, 2 \), no additional assumptions are necessary since \( W \) contractible implies it is homeomorphic to \( \mathbb{R}^n \). For \( n \geq 3 \), the existence of contractible manifolds \( W \) not homeomorphic to \( \mathbb{R}^n \) makes some restrictions likely. In the next section an example is given of a 3-dimensional, open (noncompact and empty boundary) manifold of homotopy type \( K(\Pi, 1) \) whose universal cover is not homeomorphic to \( \mathbb{R}^3 \). It is a long-standing conjecture that the universal cover of a closed (compact and empty boundary) manifold \( W^n \) of homotopy type \( K(\Pi, 1) \), \( \Pi \) an infinite group, is homeomorphic to \( \mathbb{R}^n \). Some partial results to this conjecture in the case \( n \geq 5 \) can be found (without proof) in [3]. In [7] Waldhausen gives sufficient conditions for the universal cover of a closed 3-manifold of homotopy type \( K(\Pi, 1) \) to be homeomorphic to \( \mathbb{R}^3 \). The author knows of no counterexample to this conjecture.

This paper is concerned with open \( K(\Pi, 1) \) manifolds which admit a piecewise linear structure. For the remainder of this paper let \( W^n \) denote such a manifold, and suppose further that \( W \) has a single end which is denoted by \( \infty \). An inverse sequence of groups

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is said to be essentially constant if there exists a subsequence

so that isomorphisms $\text{Im}(f'_1) \cong \text{Im}(f'_2) \cong \text{Im}(f'_3)$ are induced. Following [5], we say that $\pi_1$ is essentially constant at $\infty$ if for a sequence $K_1 \subset K_2 \subset K_3$ of compacta with $W = \bigcup \text{int } K_i$, the sequence induced by inclusion

is essentially constant. The choice of $\{K_i\}$ of base points, and of connecting base paths used to define $\delta$ does not affect this property. Also $\pi_1(\infty) = \text{proj lim}(\delta)$ proves to be independent of these choices up to isomorphism in a preferred conjugacy class; and $\pi_1(W - K_i) \rightarrow \pi_1(W)$, for large $i$, induces a homomorphism $\pi_1(\infty) \rightarrow \pi_1(W)$ which is again determined within a conjugacy class. For a verification of these facts see [6].

Let $M$ denote a compact manifold with nonempty boundary of homotopy type $K(\Pi, 1)$. If $\pi_1(\partial M)$ is isomorphic by inclusion to $\pi_1(M)$, then $W = \text{int } M$ is an open manifold with one end $\infty$, $\pi_1$ is essentially constant at $\infty$, and $\pi_1(\infty) \rightarrow \pi_1(W)$ is an isomorphism. The above properties of $W$ are, however, not sufficient to guarantee that $W$ is the interior of a compact manifold. (See [5].)

The following is the main result.

**Theorem 1.** Suppose $W^n$ is a PL manifold which has homotopy type $K(\Pi, 1)$, $W$ has one end $\infty$, $\pi_1$ is essentially constant at $\infty$, and the induced homomorphism $\pi_1(\infty) \rightarrow \pi_1(W)$ is an isomorphism. Then $W$ is PL homeomorphic to $R^n$ provided $n \geq 5$.

Before proceeding with the proof of Theorem 1, we first make an application of Theorem 1. Let $\Pi$ denote a countable group with finite cohomological dimension, denoted $\text{cd}(\Pi)$. In [3] it is shown that if $\Pi$ is countable and $\text{cd}(\Pi)$ is finite, then there is a polyhedron $X^n$ of type $K(\Pi, 1)$ of dimension $n = \text{cd}(\Pi)$ if $\text{cd}(\Pi) \neq 2$, and of dimension $n = 3$ if $\text{cd}(\Pi) = 2$. Next it is shown that $X^n$ is homotopy equivalent to a subpolyhedron $Y^n$ of $R^{2n}$. Let $W$ denote the interior of a regular neighborhood of $Y$. When $\text{cd}(\Pi) \geq 2$, $Y$ has co-dimension $\geq 3$ in $W$ and therefore $W$ has one end $\infty$, $\pi_1$ is essentially con-
stant at \( \infty \), and \( \pi_1(\infty) \to \pi_1(W) \) is an isomorphism. We have the following generalization of Corollary 3 of [3].

**Theorem 2.** Let \( \Pi \) be a countable group of finite cohomological dimension \( n \). If \( n \neq 2 \), there is a covering action of \( \Pi \) on \( R^{2n} \), and if \( n = 2 \) there is a covering action of \( \Pi \) on \( R^6 \).

**Proof.** If \( n \geq 2 \) the theorem follows from preceding remarks together with Theorem 1. If \( n = 1 \), this is Theorem 2 of [3].

All work is done in the PL category. [2] and [8] form standard references. \( \text{Int} \) and \( \partial \) are used to denote interior and boundary, respectively, a superscript is used to denote dimension and "\( \cong \)" denotes "is PL homeomorphic to".

2. First we prove two lemmas.

**Lemma 1.** Suppose \( W^n, n \geq 5 \), is a PL manifold which has homotopy type \( K(\Pi, 1) \), \( W \) has a single end \( \infty \), \( \pi_1 \) is essentially constant at \( \infty \), and the induced homomorphism \( \pi_1(\infty) \to \pi_1(W) \) is an isomorphism. Then given a compact set \( C \) in \( W \), there is a compact set \( D \) containing \( C \) such that \( \pi_i(W, W - D) = 0 \) for \( i = 0, 1, 2 \).

**Proof.** From [5, Proposition 1.9, Part A], we have the existence of an arbitrarily small connected neighborhood \( V \) of \( \infty \), such that in the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(\infty) & \xrightarrow{j_*} & \pi_1(V) \\
\downarrow{i_*} & & \downarrow{i_*} \\
\pi_1(W) & & \pi_1(W) \\
\end{array}
\]

\( j_* \) is an isomorphism by construction, \( i_* \) is assumed to be an isomorphism, hence \( i_* \) is an isomorphism. But \( \pi_2(W) = 0 \) and from the homotopy sequence of the pair \( (W, V) \), we have \( \pi_i(W, V) = 0 \) for \( i = 0, 1, 2 \). By the definition of neighborhood of \( \infty \), \( W - V \) is compact. Therefore given a compact set \( C \), let \( V \) be a connected neighborhood of \( \infty \) such that \( V \cap C = \emptyset \), and such that \( \pi_1(\infty) \to \pi_1(V) \) is an isomorphism. Then \( W - V \) is compact, contains \( C \), and \( \pi_i(W, V) = 0 \) for \( i = 0, 1, 2 \).

**Lemma 2.** Let \( K \) be a triangulation of the manifold \( W \). Let \( J \) denote
the $i$-skeleton of $K$, and $L$ the dual $(n - i - 1)$-skeleton of $K$. Let $C$ denote a compact subset of $W$ such that $C \cap L = \emptyset$. Then there exists a finite subcomplex $J_0$ of $J$ and a homeomorphism $k$ of $W$ such that

1. $k|J \cup L = 1$,
2. $k(U) \supset C$,

where $U$ denotes the simplicial neighborhood of $J_0$ in $K^n$.

Proof. As usual let $N(J, K^n)$ and $N(L, K^n)$ denote simplicial neighborhoods of $J$ and $L$, respectively, in $K^n$. Observe that since $L$ is the dual skeleton to $J$, $\partial(N(J, K^n)) = \partial(N(L, K^n))$, and from the mapping cylinder structure of derived neighborhoods,

$$N(L, K^n) - L \simeq \partial(N(L, K^n)) \times [0, 1).$$

Let

$$Y = N(J, K^n) \cup (N(L, K^n) - L).$$

But $C \cap L = \emptyset$, so $C \subset Y$, and since $C$ is compact, one can use the product structure on $N(L, K^n) - L$ to find a homeomorphism $k: Y \to Y$ with compact support such that $k(N(J, K^n)) \supset C$ and $k|J = 1$. Extend $k$ by the identity to $L$. Let

$$J_0 = \{ \sigma \in J: k^{-1}(C) \cap N(\sigma, K^n) \neq \emptyset \}.$$

The compactness of $C$ implies $J_0$ is finite and from the definition of $J_0$,

$$k(N(J_0, K^n)) \supset C.$$ 

Proof of Theorem 1. From [1] it suffices to show that given a compact set $D$ contained in $W$, $D$ is contained in the interior of an $n$-ball.

Let $K$ denote a triangulation of $W$, $\tilde{K}$ the induced triangulation of $\tilde{W}$. Let $J$ denote the $(n - 3)$-skeleton of $K$, and $L$ the dual 2-skeleton. Let $\tilde{\rho}$ denote the projection of $\tilde{W}$ onto $W$ and let $C = \rho(D)$. $C$ is compact. By Lemma 1, there is a neighborhood $V$ of $\infty$ such that

1. $C \subset W - V$,
2. $\pi_i(W, V) = 0$, $i = 0, 1, 2$.

From standard engulfing, as in [2], there is a homeomorphism $h: W \to W$ with compact support such that $h(V)$ contains $L$. Thus $h(C) \cap L = \emptyset$. Let $\tilde{h}$ denote the homeomorphism on $\tilde{W}$ which covers $h$. Then $\tilde{h}(D) \cap L = \emptyset$.

Let $D' = \tilde{h}(D)$. From Lemma 2, one can find a finite subcomplex $\tilde{J}_0$ of $\tilde{J}$ and a homeomorphism $k: \tilde{W} \to \tilde{W}$ such that $k(N(\tilde{J}_0, \tilde{K}^n)) \supset D'$. Now $\tilde{W}$ is contractible, $\dim J_0 \leq n - 3$, hence $\tilde{J}_0$ lies in the interior of a ball. $\tilde{W}$ can be collapsed to $\tilde{J}_0$, hence $N(J_0, \tilde{K}^n)$ lies in the interior of a ball. Let $B$ denote such a ball. Then $k(B) \supset D' = k(D)$ and $\tilde{W} = k(B) \supset D$. 

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The following is an outline of the construction of the 3-dimensional example promised in the introduction. Let $M^3$ denote an open $K(\Pi, 1)$ manifold, and let $W^3$ denote an open, contractible 3-manifold which does not embed in $R^3$ [4]. Let $L_1$ and $L_2$ denote the images of $[0, \infty)$ under PL embeddings in $M$ and $W$, respectively. Let $M'$ and $W'$ denote the complements in $M$ and $W$, respectively, of the interior of a regular neighborhood of each of $L_1$ and $L_2$. Then $\partial M' \cong \partial W' \cong R^2$. Let $N = M' \cup_h W'$, where $h$ is an orientation preserving homeomorphism from $\partial M'$ onto $\partial W'$. Then the universal cover $\tilde{N}$ of $N$ is an open contractible manifold and so $N$ is a $K(\Pi, 1)$ manifold. Also $W' \subset N$ lifts to infinitely many disjoint copies of $W'$ in $N$. But $\text{int } W'$ is homeomorphic to $W$ and so $N$ cannot be homeomorphic to $R^3$.

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