A CHARACTERIZATION OF HYPOELLIPTIC DIFFERENTIAL OPERATORS WITH VARIABLE COEFFICIENTS

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ABSTRACT. Let $P$ be a linear differential operator with coefficients in $C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$. We characterize the hypoelliptic operators in terms of the $*$-hypoelliptic operators. $P$ is defined to be $*$-hypoelliptic on $\Omega$ if and only if $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\infty(\Omega)$ imply $u \in C^\infty(\Omega)$. We characterize the $*$-hypoelliptic operators via a priori estimates. We prove $P$ is hypoelliptic on $\Omega$ if and only if for $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ with $\Omega' \subset \Omega$, there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of $x_0$ such that $P$ is $*$-hypoelliptic on $\Omega_{x_0}$.

Throughout this paper we will consider linear differential operators whose coefficients are in $C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$. Schwartz [8] first defined hypoelliptic operators on $\Omega \subset \mathbb{R}^n$. A differential operator $P$ is hypoelliptic on $\Omega \subset \mathbb{R}^n$ if and only if $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$, where $\Omega' \subset \Omega$, imply $u \in C^\infty(\Omega')$. Weyl [10] showed that weak solutions of Laplace's equation are in $C^\infty$. Hörmander [2] characterized the hypoelliptic operators with constant coefficients. Hörmander [3] and Malgrange [5] have shown that the formally hypoelliptic operators are hypoelliptic. This class includes the elliptic and $p$-parabolic operators with variable coefficients. Fedor [1] gives sufficient conditions which allow us to conclude that

$$P = -\frac{\partial^2}{\partial x^2} - \phi^2(x) \frac{\partial^2}{\partial y^2}$$

is hypoelliptic on $\mathbb{R}^2$ where $\phi(x) \in C^\infty(\mathbb{R}^2)$, $\phi(x) > 0$ for $x \neq 0$ and $\phi(0) = 0$.


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1 This paper is part of a doctoral dissertation written under Professor Y. W. Chen at the University of Massachusetts in Amherst.
In Theorem 6 we characterize the hypoelliptic operators with variable coefficients.

The condition of formal hypoellipticity and the sufficient conditions in Fedii [1] imply for all \( x_0 \in \Omega' \) and for any \( \Omega' \subset \Omega \) there exists a relatively compact open neighborhood \( \Omega_{x_0} \subset \Omega' \) such that for \( u \in D'_F(\Omega_{x_0}) = \bigcup_m (C_c^m(\Omega_{x_0}))' \) and \( Pu \in C^\infty(\Omega_{x_0}) \) imply \( u \in C^\infty(\Omega_{x_0}) \). Operators with this property are hypoelliptic. Hence we define \(*\)-hypoellipticity on \( \Omega \).

**Definition.** \( P \) is \(*\)-hypoelliptic on \( \Omega \) if and only if \( u \in L^p(\Omega) \) and \( Pu \in C^\infty(\Omega) \) imply \( u \in C^\infty(\Omega) \).

First, in Theorem 3, we characterize \(*\)-hypoelliptic operators on \( \Omega \), and then in Theorem 6 we characterize hypoelliptic operators via \(*\)-hypoelliptic operators on neighborhoods of elements in \( \Omega \). For the constant coefficient case, hypoellipticity is characterized algebraically in terms of the differential polynomial. The characterization of \(*\)-hypoelliptic operators will be in terms of a priori estimates one of which is similar to the subelliptic estimates for elliptic operators.

In order to characterize \(*\)-hypoelliptic operators on \( \Omega \), we need a theorem about linear operators on subspaces of \( D'(\Omega) \). The motivation for this theorem comes from a discussion in Trèves [9, p. 538] of local fundamental solutions of an operator whose transpose is hypoelliptic.

**Theorem 1.** Let \( E, F, G \) be Fréchet spaces such that \( D(\Omega) \subset E, F, G \subset D'(\Omega) \), the inclusion maps are continuous and the topologies are given by sequences of seminorms \((p_n), (q_n)\) and \((g_n)\) respectively. Also let \( Q, P : E \to D'(\Omega) \) be continuous linear maps and define \( R = \{ u \in E | Pu \in F \} \). If \( Q : R \to G \) is a linear map, then \( R \) may be made into the same Fréchet space by either of the sequences of seminorms

\[
\alpha_n(u) = p_n(u) + q_n(Pu) + g_n(Qu) \quad \text{and} \quad \gamma_n(u) = p_n(u) + q_n(Pu).
\]

**Proof.** Since \( P : E \to D'(\Omega) \) is linear, \( R \) is a vector space. Since \( Q : R \to G \), both \( \alpha_n \) and \( \gamma_n \) are well defined. Since \( P \) and \( Q \) are linear maps, \( \alpha_n \) and \( \gamma_n \) are seminorms. If the graph \( R \times PR \) is closed in \( E \times F \), the locally convex metric space \( (R, (\gamma_n)) \) will be complete and hence a Fréchet space. Let \( u_n \in R, u_n \to u \in E \) and \( Pu_n \to v \) in \( F \). Since \( P : E \to D'(\Omega) \) is continuous, \( Pu_n \to Pu \) in \( D'(\Omega) \). Since \( Pu_n \to v \) in \( F \), \( Pu_n \to v \) in \( D'(\Omega) \). Thus \( Pu = v \in F \) and \( u \in R \).

In order to show \( (R, (\alpha_n)) \) is complete, let \( (u_n) \) be Cauchy in \( (R, (\alpha_n)) \). Since \( (R, (\gamma_n)) \) and \( G \) are complete, there exist \( u \in R \) and \( v \in G \) such that
In the next theorem we establish necessary conditions for *-hypoellipticity. These conditions will be a priori estimates on subspaces of $H^{10c}(\Omega)$. Recall $H_s = \{u \in \mathcal{S}'|\hat{u}(\xi) = \text{the Fourier transform of } u(x) \text{ is a function and } \hat{u}(\xi)(1 + |\xi|^2)^{s/2} \in L_2\}$. With the norm $\|u\|_s = \|\hat{u}(\xi)(1 + |\xi|^2)^{s/2}\|_{L_2}$, $H_s$ is a Banach space. $H^c_s(K) = \{u \in H_s|\text{support } u \subset K, K \text{ compact}\}$ is a closed subspace of $H_s$ and hence a Banach space. Let

$$\text{Theorem 2. If } P \text{ is *-hypoelliptic on } \Omega, \text{ then the following hold for all } s \in \mathbb{R};$$

(A) For all $\phi_n$ there exist $\phi_m$ and $C = C(n, s) > 0$ such that

$$\|\phi_n u\|_s \leq C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})$$

where $u \in \{u \in H^{10c}_{s-1}(\Omega)|Pu \in C^\infty(\Omega)\}$.

(B) For all compact $K \subset \Omega$ there exist $m$ and $C = C(K, s) > 0$ such that

$$\|u\|_s \leq C(\|Pu\|_m + \|u\|_{s-1})$$

where $u \in \{u \in H^c_{s-1}(K)|Pu \in C^\infty(K)\}$.

(C) For all $\phi_n$ and $\phi \in C^\infty_c(\Omega)$ there exist $\phi_m$ and $C = C(n, \phi, s) > 0$ such that
\[ \| \phi_n [P, \phi] u \|_n \leq C(\| \phi_m Pu \|_m + \| \phi_m u \|_{s-1}) \]

where \( u \in \{ u \in H_{s-1}^{1,0}(\Omega) | Pu \in C^\infty(\Omega) \} \).

**Proof.** In order to prove estimate (A) apply Theorem 1 with \( E = H_{s-1}^{1,0}(\Omega) \), \( F = C^\infty(\Omega) \), \( G = H_{s-1}^{1,0}(\Omega) \), \( p_n(u) = \| \phi_n u \|_{s-1} \), \( q_n(u) = \| \phi_n u \|_n \), \( b_n(u) = \| \phi_n u \|_s \), \( P = P \) and \( Q \) the inclusion map \( H_{s-1}^{1,0}(\Omega) \to \mathcal{D}'(\Omega) \). The assumption that \( P \)

\[ R = \{ u \in H_{s-1}^{1,0}(\Omega) | Pu \in C^\infty(\Omega) \} \subset H_{s-1}^{1,0}(\Omega) = G. \]

From the conclusion of Theorem 1 we have for all \( n \) there exist \( m \) and a constant \( C > 0 \) such that \( \alpha_n(u) \leq Cy(u) \) for all \( u \in R \). This gives our desired estimate.

Estimate (B) follows from estimate (A) for a suitable choice of \( \phi_n \).

This estimate may also be proved by a direct application of Theorem 1.

In order to prove estimate (C), let \( E = H_{s-1}^{1,0}(\Omega) \), \( F = C^\infty(\Omega) \), \( P = P \) and \( Q \) the map from \( H_{s-1}^{1,0}(\Omega) \to \mathcal{D}'(\Omega) \) given by \( u \mapsto [P, \phi] u \). The assumption in Theorem 1 that \( R = \{ u \in H_{s-1}^{1,0}(\Omega) | Pu \in C^\infty(\Omega) \} \subset C^\infty(\Omega) = G \) follows from the \( \ast \)-hypoellipticity on \( \Omega \). By the conclusion of Theorem 1 we obtain estimate (C).

**Theorem 3.** The following are equivalent:

1. \( P \) is \( \ast \)-hypoelliptic on \( \Omega \);
2. Estimate (A) holds for all \( s \in R \);
3. Estimates (B) and (C) hold for all \( s \in R \).

**Proof.** Theorem 2 gives 1 \( \Rightarrow \) 2 and 1 \( \Rightarrow \) 3. First we will show 2 \( \Rightarrow \) 1.

Let \( u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega) \). Since \( u \in \mathcal{D}'(\Omega) \), \( u \in H_{s-1}^{1,0}(\Omega) \) for some \( s \in R \). Apply estimate (A) for this \( s \in R \). This gives for all \( \phi_n, \| \phi_n u \|_s < \infty \) and hence \( u \in H_{s}^{1,0}(\Omega) \). Iterate this to get \( u \in \bigcap_s H_{s}^{1,0}(\Omega) = C^\infty(\Omega) \).

In order to prove 3 \( \Rightarrow \) 1, let \( u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega) \). Since \( u \in H_{s}^{1,0}(\Omega) \) for some \( s \in R \), estimate (C) implies for suitable choice of \( \phi_n \) that \( \| [P, \phi] u \|_n < \infty \). Since \( [P, \phi] u = \phi Pu - Pu(\phi u), \| Pu(\phi u) \|_n < \infty \) for large \( n \). In order to obtain \( \| \phi u \|_s < \infty \), use estimate (B) with \( u \) replaced by \( \phi u \). This holds for all \( \phi \in C_c^\infty(\Omega) \) and hence \( u \in H_{s}^{1,0}(\Omega) \). Iterate this to get \( u \in \bigcap_{s \in R} H_{s}^{1,0}(\Omega) = C^\infty(\Omega) \).

If \( P \) is an elliptic operator, then one may show \( \| u \|_{s+m} \leq C(\| Pu \|_s + \| u \|_{s+m-1}) \), where \( m = \text{order } P \), \( u \in C^\infty_c(K) \) with \( K \subset \Omega \) compact.

Such estimates are called subelliptic. Estimate (B) is similar to subelliptic estimates and, in fact, subelliptic estimates may be extended from \( C_c^\infty(K) \) to
\[ u \in H_{s+1}^{c}(K) \mid Pu \in H_{s}^{c}(K) \]. See Nieto [6] or Peetre [7] for a proof of this.

It is interesting to note that in order to establish *-hypoellipticity on \( \Omega \) for elliptic operators, we need only the extended subelliptic estimate and not estimate (C). If \( u \in \{ u \mid u \in H_{s+1}^{loc}(\Omega) \mid Pu \in H_{s}^{loc}(\Omega) \} \), then \([P, \phi]u \in H_{s}^{loc}(\Omega)\) since the order of \([P, \phi]\) is \( m - 1 \). Since \([P, \phi]u = P(\phi u) - \phi Pu, P(\phi u) \in H_{s}^{c}(K)\) where support \( \phi \subseteq K \). Apply the extended subelliptic estimate to conclude \( u \in H_{s+1}^{loc}(\Omega) \). Iterate this to conclude that elliptic operators are *-hypoelliptic.

**Definition.** \( P \) is local on \( \Omega \) if and only if \( u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega') \) with \( \Omega' \subseteq \Omega \) imply \( P(\phi u) \in C^\infty(\Omega') \) for all \( \phi \in C^\infty(\Omega') \). \( P \) is *-*local on \( \Omega \) if and only if \( u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega) \) imply \( P(\phi u) \in C^\infty(\Omega) \) for all \( \phi \in C^\infty(\Omega) \).

If for all \( \Omega' \subseteq \Omega \) such that \( u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega') \) there exists for each \( x_0 \in \Omega' \) a relatively compact open neighborhood \( \Omega_{x_0} \subseteq \Omega' \) of \( x_0 \) with \( P \)-local on \( \Omega_{x_0} \), then \( P \) is local on \( \Omega \). In order to show this, let \( \phi \in C^\infty_c(\Omega'), u \in \mathcal{D}'(\Omega) \) and \( Pu \in C^\infty(\Omega') \). Let \((\psi_i)\) be a partition of unity subordinate to the finite subcover \((\Omega_{x_i})_i \subseteq \text{support } \phi \) of \( \text{support } \phi \) which is compact.

\[
P(\phi u) = P \left( \sum_{i=1}^{M<\infty} \psi_i \phi u \right) = \sum_{i=1}^{M<\infty} P(\psi_i \phi u).
\]

Since \( P \) is *-*local on each \( \Omega_{x_0} \), \( P \) is *-*local on \( \Omega_{x_i} \). Since \( \Omega_{x_i} \) are relatively compact, \( u \in \mathcal{D}'(\Omega_{x_i}) \) and \( Pu \in C^\infty(\Omega_{x_i}) \). Hence \( P(\psi_i \phi u) \in C^\infty(\Omega_{x_i}) \) and \( P(\phi u) \in C^\infty(\Omega') \). We next characterize the *-*local operators.

**Theorem 4.** The following are equivalent:

1. \( P \) is *-*local on \( \Omega \).

2. For all \( s \in \mathbb{R}, \phi_n \in C^\infty_c(\Omega), \phi \in C^\infty_c(\Omega), \) there exist \( \phi_m \) and \( C = C(s, \phi_n, \phi) > 0 \) such that

\[
\|\phi_n P(\phi u)\|_n \leq C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})
\]

where \( u \in \{u \mid u \in H_{s+1}^{loc}(\Omega) \mid Pu \in C^\infty(\Omega)\} \).

3. For all \( s \in \mathbb{R}, \phi_n \in C^\infty_c(\Omega), \phi \in C^\infty_c(\Omega), \) there exist \( \phi_m \) and \( C = C(s, \phi_n, \phi) > 0 \) such that

\[
\|\phi_n [P, \phi]u\|_n \leq C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})
\]

where \( u \in \{u \mid u \in H_{s+1}^{loc}(\Omega) \mid Pu \in C^\infty(\Omega)\} \).
Proof. 1 → 2 is a consequence of Theorem 1. Let $E = H_{s-1}^{10c}(\Omega)$, $F = C^\infty(\Omega)$, $G = C^\infty(\Omega)$, $P = P$ and $Q$ the map given by $u \mapsto P(\phi u)$. The fact that $Q: R = \{ u \in H_{s-1}^{10c}(\Omega) | Pu \in C^\infty(\Omega) \} \to G$ follows from $P$ being $*$-local on $\Omega$. 2 → 1 follows from the inequality in 2 and the fact that for large $n$, $\phi_n P(\phi u) = P(\phi u)$. In order to show 1 → 3, note by $P(\phi u) = \phi Pu + [P, \phi] u$, $P$ is $*$-local on $\Omega$ if and only if $u \in D'_F(\Omega)$ and $Pu \in C^\infty$ imply $[P, \phi] u \in C^\infty(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$. Apply Theorem 1 as above, except let $Q$ be given by the map $u \mapsto [P, \phi] u$.

The next theorem is concerned with extensions in some sense of certain distributions on $U \subset \Omega$ to distributions on $\Omega$. This theorem allows us to characterize both the hypoelliptic and local operators on $\Omega$.

**Theorem 5.** Let $U' \subset U \subset \Omega$ be open subsets in $\mathbb{R}^n$ such that $\overline{U'}$ is a compact subset of $U$. If $u \in D'_F(U)$, then there exists $v \in D'_F(\Omega)$ such that $v|_{\overline{U'}} = u|_{\overline{U'}}$.

**Proof.** Since $u \in D'_F(U)$, there exists an integer $m$ such that $u \in (C^m_c(U))'$. Since the inclusion map $C^m_c(U') \to C^m_c(U)$ is continuous, $u|_{\overline{U'}}$ is continuous on $C^m_c(\Omega)$. Thus there exists a constant $C > 0$ such that

$$|(u|_{\overline{U'}}, \phi)| \leq C \sup_x \sum_{|a| \leq m} |D^a \phi(x)|$$

for all $\phi \in C^\infty_c(U')$. Let $\psi_\alpha \in C_c^\infty(\Omega)$ be such that $\psi_\alpha \equiv 1$ on $\overline{U'}$ and $|\alpha| \leq m$. Define

$$p(\phi) = \sup_x \sum_{|\alpha| \leq m} |\psi_\alpha(x)D^\alpha \phi(x)|$$

where $\phi \in C_c^\infty(\Omega)$. Since $(\psi_\alpha)$ is a finite family, and hence locally finite on $\Omega$, $p(\phi)$ is a seminorm from the usual topology on $C_c^\infty(\Omega)$. See Hörmander [4, p. 6] for what is essentially the proof of this. Thus $|(u|_{\overline{U'}}, \phi)| \leq p(\phi)$ for all $\phi \in C^\infty_c(U')$, which is a closed subspace of $C^\infty_c(\Omega)$. By the Hahn-Banach theorem there exists $v \in D'_F(\Omega)$ such that $v|_{\overline{U'}} = u|_{\overline{U'}}$.

**Theorem 6.** $P$ is hypoelliptic on $\Omega$ if and only if for all $\Omega' \subset \Omega$ with $u \in \tilde{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega')$ there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of $x_0$ such that $P$ is $*$-hypoelliptic on $\Omega_{x_0}$.

**Proof.** Let $P$ be hypoelliptic on $\Omega$. Let $\Omega' \subset \Omega$ be any open subset. Choose $\Omega_{x_0} \subset \Omega'$ to be any relatively compact open neighborhood of $x_0$. Assume $u \in \tilde{D}'_F(\Omega_{x_0})$ and $Pu \in C^\infty(\Omega_{x_0})$. In order to show $u \in C^\infty(\Omega_{x_0})$, we need to show...
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we must find a neighborhood $\Omega_y \subset \Omega_{x_0}$ of each $y \in \Omega_{x_0}$ such that $u \in C^\infty(\Omega_y)$. Let $\Omega_y$ be any relatively compact neighborhood of $y \in \Omega_{x_0}$ such that $\Omega_y \subset \Omega_{x_0}$. By Theorem 5 there exists $v \in \mathcal{D}'(\Omega)$ such that $v|_{\Omega_y} = u|_{\Omega_y}$. Since $Pu \in C^\infty(\Omega_{x_0})$, $Pu = Pv \in C^\infty(\Omega_y)$. Since $P$ is hypoelliptic on $\Omega$, $u = v|_{\Omega_y} \in C^\infty(\Omega_y)$.

Suppose $P$ is $*$-hypoelliptic on each $\Omega_{x_0}$ and let $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$. We must show $u \in C^\infty(\Omega_{x_0})$ for all the given $\Omega_{x_0} \subset \Omega'$. Since $\Omega_{x_0}$ is relatively compact, $u \in \mathcal{D}'(\Omega_{x_0})$. Also $Pu \in C^\infty(\Omega_{x_0})$, and since $P$ is $*$-hypoelliptic on $\Omega_{x_0}$, $u \in C^\infty(\Omega_{x_0})$.

Theorem 7. $P$ is local on $\Omega$ if and only if for all $\Omega' \subset \Omega$ with $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of $x_0 \in \Omega'$ such that $P$ is $*$-local on $\Omega_{x_0}$.

Proof. Let $P$ be local on $\Omega$. Let $\Omega' \subset \Omega$ be any open subset and choose $\Omega_{x_0} \subset \Omega'$ to be any relatively compact open neighborhood of $x_0 \in \Omega'$. Assume $u \in \mathcal{D}'(\Omega_{x_0})$ and $Pu \in C^\infty(\Omega_{x_0})$. Choose $(\Omega_{y_i})$ to be an open locally finite cover of $\Omega_{x_0}$ such that $\bigcup_{i} \Omega_{y_i} = \Omega_{x_0}$. By Theorem 5 there exists $v \in \mathcal{D}'(\Omega)$ such that $v|_{\Omega_{y_i}} = u|_{\Omega_{y_i}}$. Since $Pu = P v \in C^\infty(\Omega_{y_i})$ and $P$ is local on $\Omega$, $P(\phi y_i u) \in C^\infty(\Omega_{y_i})$, where $\psi \in C^\infty_c(\Omega_{y_i})$. Let $(\psi_i)$ be a partition of unity subordinate to $(\Omega_{y_i})$.

$$P(\phi y_i u) = \left( \sum_{i=1}^{M<\infty} \psi_i \phi y_i u \right) = \sum_{i=1}^{M<\infty} P(\psi_i \phi y_i u) \in C^\infty(\Omega_{x_0}).$$

The proof of the converse is given after the definitions of local and $*$-local operators on $\Omega$.

Theorem 7 gives an interpretation of estimates (C) in Theorem 2. Estimates (B) in Theorem 2 imply the existence of local fundamental solutions for the transpose of $P$. For the proof of this see Trèves [9]. In fact, it is easy to prove that hypoelliptic operators on $\Omega$ are characterized by the existence of local fundamental solutions, which may be extended continuously to $C^\infty(\Omega_{x_0})$, of the transpose of $P$ and by $P$ being local on $\Omega$.

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