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A CHARACTERIZATION OF HYPOELLIPTIC DIFFERENTIAL OPERATORS WITH VARIABLE COEFFICIENTS

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ABSTRACT. Let P be a linear differential operator with coefficients in $C^\infty(\Omega)$ where $\Omega \subset \mathbf{R}^n$. We characterize the hypoelliptic operators in terms of the $*$-hypoelliptic operators. P is defined to be $*$-hypoelliptic on Ω if and only if $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega)$ imply $u \in C^\infty(\Omega)$. We characterize the $*$-hypoelliptic operators via a priori estimates. We prove P is hypoelliptic on Ω if and only if for $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ with $\Omega' \subset \Omega$, there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of x_0 such that P is $*$-hypoelliptic on Ω_{x_0}.

Throughout this paper we will consider linear differential operators whose coefficients are in $C^\infty(\Omega)$ where $\Omega \subset \mathbf{R}^n$. Schwartz [8] first defined hypoelliptic operators on $\Omega \subset \mathbf{R}^n$. A differential operator P is *hypoelliptic on* $\Omega \subset \mathbf{R}^n$ if and only if $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$, where $\Omega' \subset \Omega$, imply $u \in C^\infty(\Omega')$. Weyl [10] showed that weak solutions of Laplace's equation are in C^∞. Hörmander [2] characterized the hypoelliptic operators with constant coefficients. Hörmander [3] and Malgrange [5] have shown that the formally hypoelliptic operators are hypoelliptic. This class includes the elliptic and p-parabolic operators with variable coefficients. Fediĭ [1] gives sufficient conditions which allow us to conclude that

$$P = -\frac{\partial^2}{\partial x^2} - \phi^2(x)\frac{\partial^2}{\partial y^2}$$

is hypoelliptic on $\mathbf{R}^2$ where $\phi(x) \in C^\infty(\mathbf{R}^2)$, $\phi(x) > 0$ for $x \neq 0$ and $\phi(0) =$

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0. In Theorem 6 we characterize the hypoelliptic operators with variable coefficients.

The condition of formal hypoellipticity and the sufficient conditions in Fediĭ [1] imply for all $x_0 \in \Omega'$ and for any $\Omega' \subset \Omega$ there exists a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of x_0 such that for $u \in \mathfrak{D}'_F(\Omega_{x_0}) = \bigcup_m (C_c^m(\Omega_{x_0}))'$ and $Pu \in C^\infty(\Omega_{x_0})$ imply $u \in C^\infty(\Omega_{x_0})$. Operators with this property are hypoelliptic. Hence we define $*$-hypoellipticity on Ω.

Definition. P is *$*$-hypoelliptic on* Ω if and only if $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega)$ imply $u \in C^\infty(\Omega)$.

First, in Theorem 3, we characterize $*$-hypoelliptic operators on Ω, and then in Theorem 6 we characterize hypoelliptic operators via $*$-hypoelliptic operators on neighborhoods of elements in Ω. For the constant coefficient case, hypoellipticity is characterized algebraically in terms of the differential polynomial. The characterization of $*$-hypoelliptic operators will be in terms of a priori estimates one of which is similar to the subelliptic estimates for elliptic operators.

In order to characterize $*$-hypoelliptic operators on Ω, we will need a theorem about linear operators on subspaces of $\mathfrak{D}'(\Omega)$. The motivation for this theorem comes from a discussion in Trèves [9, p. 538] of local fundamental solutions of an operator whose transpose is hypoelliptic.

Theorem 1. *Let E, F, G be Fréchet spaces such that $\mathfrak{D}(\Omega) \subset E, F, G \subset \mathfrak{D}'(\Omega)$, the inclusion maps are continuous and the topologies are given by sequences of seminorms (p_n), (q_n) and (g_n) respectively. Also let Q, P: $E \to \mathfrak{D}'(\Omega)$ be continuous linear maps and define $R = \{u \in E | Pu \in F\}$. If Q: $R \to G$ is a linear map, then R may be made into the same Fréchet space by either of the sequences of seminorms*

$$\alpha_n(u) \equiv p_n(u) + q_n(Pu) + g_n(Qu) \quad \textit{and} \quad \gamma_n(u) \equiv p_n(u) + q_n(Pu).$$

Proof. Since P: $E \to \mathfrak{D}'(\Omega)$ is linear, R is a vector space. Since Q: $R \to G$, both α_n and γ_n are well defined. Since P and Q are linear maps, α_n and γ_n are seminorms. If the graph $R \times PR$ is closed in $E \times F$, the locally convex metric space $(R, (\gamma_n))$ will be complete and hence a Fréchet space. Let $u_n \in R$, $u_n \to u \in E$ and $Pu_n \to v$ in F. Since P: $E \to \mathfrak{D}'(\Omega)$ is continuous, $Pu_n \to Pu$ in $\mathfrak{D}'(\Omega)$. Since $Pu_n \to v$ in F, $Pu_n \to v$ in $\mathfrak{D}'(\Omega)$. Thus $Pu = v \in F$ and $u \in R$.

In order to show $(R, (\alpha_n))$ is complete, let (u_n) be Cauchy in $(R, (\alpha_n))$. Since $(R, (\gamma_n))$ and G are complete, there exist $u \in R$ and $v \in G$ such that

$u_n \to u$ in E, $Pu_n \to Pu$ in F and $Qu_n \to v$ in G. Since Q is continuous on E, $Qu_n \to Qu$ in $\mathfrak{D}'(\Omega)$. Also $Qu_n \to v$ in $\mathfrak{D}'(\Omega)$. Thus $Qu = v \in G$ and $(R, (\alpha_n))$ is a Fréchet space. Since the identity map from $(R, (\alpha_n)) \to (R, (\gamma_n))$ is continuous, we may apply the open mapping theorem in order to get our desired conclusion.

In the next theorem we establish necessary conditions for $*$-hypoellipticity. These conditions will be a priori estimates on subspaces of $H_s^{\mathrm{loc}}(\Omega)$. Recall $H_s = \{u \in \mathcal{S}' | \hat{u}(\xi) =$ the Fourier transform of $u(x)$ is a function and $\hat{u}(\xi)(1 + |\xi|^2)^{s/2} \in L_2\}$. With the norm $\|u\|_s = \|\hat{u}(\xi)(1 + |\xi|^2)^{s/2}\|_{L_2}$, H_s is a Banach space. $H_s^c(K) = \{u \in H_s | \text{support } u \subset K, K \text{ compact}\}$ is a closed subspace of H_s and hence a Banach space.

$$H_s^{\mathrm{loc}}(\Omega) = \{u \in \mathfrak{D}'(\Omega) | \phi u \in H_s \text{ for all } \phi \in C_c^\infty(\Omega)\}$$

is topologized via the sequence of seminorms $p_n(u) = \|\phi_n u\|_s$, where $\phi_n \in C_c^\infty(\Omega)$, support $\phi_n \subset$ support ϕ_{n+1}, $\phi_{n+1} \equiv 1$ on support ϕ_n and $\bigcup_n$ support $\phi_n = \Omega$. $H_s^{\mathrm{loc}}(\Omega)$ is complete and hence a Fréchet space. For the proofs of these facts see Hörmander [4]. Let

$$P^\alpha(\xi) = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}} P(\xi) = i^{|\alpha|} D^\alpha P(\xi)$$

and

$$[P, \phi]u = P(\phi u) - \phi P u = \sum_{0<|\alpha|} \frac{D^\alpha \phi}{\alpha!} \cdot P^\alpha u.$$

Theorem 2. *If P is $*$-hypoelliptic on Ω, then the following hold for all $s \in \mathbf{R}$:*

(A) *For all ϕ_n there exist ϕ_m and $C = C(n, s) > 0$ such that*

$$\|\phi_n u\|_s \le C(\|\phi_m P u\|_m + \|\phi_m u\|_{s-1})$$

where $u \in \{u \in H_{s-1}^{\mathrm{loc}}(\Omega) | Pu \in C^\infty(\Omega)\}$.

(B) *For all compact $K \subset \Omega$ there exist m and $C = C(K, s) > 0$ such that*

$$\|u\|_s \le C(\|Pu\|_m + \|u\|_{s-1})$$

where $u \in \{u \in H_{s-1}^c(K) | Pu \in C_c^\infty(K)\}$.

(C) *For all ϕ_n and $\phi \in C_c^\infty(\Omega)$ there exist ϕ_m and $C = C(n, \phi, s) > 0$ such that*

$$\|\phi_n[P, \phi]u\|_n \leq C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})$$

where $u \in \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) | Pu \in C^\infty(\Omega)\}$.

Proof. In order to prove estimate (A) apply Theorem 1 with $E = H^{\mathrm{loc}}_{s-1}(\Omega)$, $F = C^\infty(\Omega)$, $G = H^{\mathrm{loc}}_s(\Omega)$, $p_n(u) = \|\phi_n u\|_{s-1}$, $q_n(u) = \|\phi_n u\|_n$, $g_n(u) = \|\phi_n u\|_s$, $P = P$ and Q the inclusion map $H^{\mathrm{loc}}_{s-1}(\Omega) \to \mathfrak{D}'(\Omega)$. The assumption that P is $*$-hypoelliptic on Ω and the fact that $H^{\mathrm{loc}}_{s-1}(\Omega) \subset \mathfrak{D}'_F(\Omega)$ yields

$$R = \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) | Pu \in C^\infty(\Omega)\} \subset H^{\mathrm{loc}}_s(\Omega) = G.$$

From the conclusion of Theorem 1 we have for all n there exist m and a constant $C > 0$ such that $\alpha_n(u) \leq C\gamma_m(u)$ for all $u \in R$. This gives our desired estimate.

Estimate (B) follows from estimate (A) for a suitable choice of ϕ_n. This estimate may also be proved by a direct application of Theorem 1.

In order to prove estimate (C), let $E = H^{\mathrm{loc}}_{s-1}(\Omega)$, $F = C^\infty(\Omega)$, $P = P$ and Q the map from $H^{\mathrm{loc}}_{s-1}(\Omega) \to \mathfrak{D}'(\Omega)$ given by $u \mapsto [P, \phi]u$. The assumption in Theorem 1 that $R = \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) | Pu \in C^\infty(\Omega)\} \subset C^\infty(\Omega) = G$ follows from the $*$-hypoellipticity on Ω. By the conclusion of Theorem 1 we obtain estimate (C).

Theorem 3. *The following are equivalent*:

1. *P is $*$-hypoelliptic on Ω.*
2. *Estimate* (A) *holds for all* $s \in \mathbf{R}$.
3. *Estimates* (B) *and* (C) *hold for all* $s \in \mathbf{R}$.

Proof. Theorem 2 gives $1 \Rightarrow 2$ and $1 \Rightarrow 3$. First we will show $2 \Rightarrow 1$. Let $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega)$. Since $u \in \mathfrak{D}'_F(\Omega)$, $u \in H^{\mathrm{loc}}_{s-1}(\Omega)$ for some $s \in \mathbf{R}$. Apply estimate (A) for this $s \in \mathbf{R}$. This gives for all ϕ_n, $\|\phi_n u\|_s < \infty$ and hence $u \in H^{\mathrm{loc}}_s(\Omega)$. Iterate this to get $u \in \bigcap_s H^{\mathrm{loc}}_s(\Omega) = C^\infty(\Omega)$.

In order to prove $3 \Rightarrow 1$, let $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega)$. Since $u \in H^{\mathrm{loc}}_{s-1}(\Omega)$ for some $s \in \mathbf{R}$, estimate (C) implies for suitable choice of ϕ_n that $\|[P, \phi]u\|_n < \infty$. Since $[P, \phi]u = \phi Pu - P(\phi u)$, $\|P(\phi u)\|_n < \infty$ for large n. In order to obtain $\|\phi u\|_s < \infty$, use estimate (B) with u replaced by ϕu. This holds for all $\phi \in C^\infty_c(\Omega)$ and hence $u \in H^{\mathrm{loc}}_s(\Omega)$. Iterate this to get $u \in \bigcap_{s \in \mathbf{R}} H^{\mathrm{loc}}_s(\Omega) = C^\infty(\Omega)$.

If P is an elliptic operator, then one may show $\|u\|_{s+m} \leq C(\|Pu\|_s + \|u\|_{s+m-1})$, where m = order P, $u \in C^\infty_c(K)$ with $K \subset \Omega$ compact. Such estimates are called subelliptic. Estimate (B) is similar to subelliptic estimates and, in fact, subelliptic estimates may be extended from $C^\infty_c(K)$ to

$\{u \in H^c_{s+m-1}(K) | Pu \in H^c_s(K)\}$. See Nieto [6] or Peetre [7] for a proof of this. It is interesting to note that in order to establish $*$-hypoellipticity on Ω for elliptic operators, we need only the extended subelliptic estimate and not estimate (C). If $u \in \{u \in H^{\mathrm{loc}}_{s+m-1}(\Omega) | Pu \in H^{\mathrm{loc}}_s(\Omega)\}$, then $[P, \phi]u \in H^{\mathrm{loc}}_s(\Omega)$ since the order of $[P, \phi]$ is $m-1$. Since $[P, \phi]u = P(\phi u) - \phi Pu$, $P(\phi u) \in H^c_s(K)$ where support $\phi \subset K$. Apply the extended subelliptic estimate to conclude $u \in H^{\mathrm{loc}}_{s+m}(\Omega)$. Iterate this to conclude that elliptic operators are $*$-hypoelliptic.

Definition. P is *local on* Ω if and only if $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ with $\Omega' \subset \Omega$ imply $P(\phi u) \in C^\infty(\Omega')$ for all $\phi \in C^\infty_c(\Omega')$. P is *$*$-local on* Ω if and only if $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty(\Omega)$ imply $P(\phi u) \in C^\infty(\Omega)$ for all $\phi \in C^\infty_c(\Omega)$.

If for all $\Omega' \subset \Omega$ such that $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of x_0 with P $*$-local on Ω_{x_0}, then P is local on Ω. In order to show this, let $\phi \in C^\infty_c(\Omega')$, $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$. Let (ψ_i) be a partition of unity subordinate to the finite subcover $(\Omega_{x_i})x_i \in$ support ϕ of support ϕ which is compact.

$$P(\phi u) = P\left(\sum_{i=1}^{M<\infty} \psi_i \phi u\right) = \sum_{i=1}^{M<\infty} P(\psi_i \phi u).$$

Since P is $*$-local on each Ω_{x_0}, P is $*$-local on Ω_{x_i}. Since Ω_{x_i} are relatively compact, $u \in \mathfrak{D}'_F(\Omega_{x_i})$ and $Pu \in C^\infty(\Omega_{x_i})$. Hence $P(\psi_i \phi u) \in C^\infty(\Omega_{x_i})$ and $P(\phi u) \in C^\infty(\Omega')$. We next characterize the $*$-local operators.

Theorem 4. *The following are equivalent:*

1. *P is $*$-local on Ω.*

2. *For all $s \in \mathbf{R}$, $\phi_n \in C^\infty_c(\Omega)$, and $\phi \in C^\infty_c(\Omega)$, there exist ϕ_m and $C = C(s, \phi_n, \phi) > 0$ such that*

$$\|\phi_n P(\phi u)\|_n \le C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})$$

where $u \in \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) | Pu \in C^\infty(\Omega)\}$.

3. *For all $s \in \mathbf{R}$, $\phi_n \in C^\infty_c(\Omega)$, and $\phi \in C^\infty_c(\Omega)$, there exist ϕ_m and $C = C(s, \phi_n, \phi) > 0$ such that*

$$\|\phi_n [P, \phi]u\|_n \le C(\|\phi_m Pu\|_m + \|\phi_m u\|_{s-1})$$

where $u \in \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) | Pu \in C^\infty(\Omega)\}$.

Proof. $1 \Rightarrow 2$ is a consequence of Theorem 1. Let $E = H^{\mathrm{loc}}_{s-1}(\Omega)$, $F = C^\infty(\Omega)$, $G = C^\infty(\Omega)$, $P = P$ and Q the map given by $u \mapsto P(\phi u)$. The fact that $Q\colon R = \{u \in H^{\mathrm{loc}}_{s-1}(\Omega) \mid Pu \in C^\infty(\Omega)\} \to G$ follows from P being $*$-local on Ω. $2 \Rightarrow 1$ follows from the inequality in 2 and the fact that for large n, $\phi_n P(\phi u) = P(\phi u)$. In order to show $1 \Leftrightarrow 3$, note by $P(\phi u) = \phi Pu + [P, \phi]u$, P is $*$-local on Ω if and only if $u \in \mathfrak{D}'_F(\Omega)$ and $Pu \in C^\infty$ imply $[P, \phi]u \in C^\infty(\Omega)$ for all $\phi \in C^\infty_c(\Omega)$. Apply Theorem 1 as above, except let Q be given by the map $u \mapsto [P, \phi]u$.

The next theorem is concerned with extensions in some sense of certain distributions on $U \subset \Omega$ to distributions on Ω. This theorem allows us to characterize both the hypoelliptic and local operators on Ω.

Theorem 5. *Let $U' \subset U \subset \Omega$ be open subsets in $\mathbf{R}^n$ such that $\overline{U'}$ is a compact subset of U. If $u \in \mathfrak{D}'_F(U)$, then there exists $v \in \mathfrak{D}'(\Omega)$ such that $v|_{\overline{U'}} = u|_{\overline{U'}}$.*

Proof. Since $u \in \mathfrak{D}'_F(U)$, there exists an integer m such that $u \in (C^m_c(U))'$. Since the inclusion map $C^m_c(\overline{U'}) \to C^m_c(U)$ is continuous, $u|_{\overline{U'}}$ is continuous on $C^m_c(\overline{U'})$. Thus there exists a constant $C > 0$ such that

$$|(u|_{\overline{U'}}, \phi)| \le C \sup_x \sum_{|\alpha| \le m} |D^\alpha \phi(x)| \quad \text{for all } \phi \in C^\infty_c(\overline{U'}).$$

Let $\psi_\alpha \in C^\infty_c(\Omega)$ be such that $\psi_\alpha \equiv C$ on $\overline{U'}$ and $|\alpha| \le m$. Define

$$p(\phi) = \sup_x \sum_{|\alpha| \le m} |\psi_\alpha(x) D^\alpha \phi(x)| \quad \text{where } \phi \in C^\infty_c(\Omega).$$

Since (ψ_α) is a finite family, and hence locally finite on Ω, $p(\phi)$ is a seminorm from the usual topology on $C^\infty_c(\Omega)$. See Hörmander [4, p. 6] for what is essentially the proof of this. Thus $|(u|_{\overline{U'}}, \phi)| \le p(\phi)$ for all $\phi \in C^\infty_c(\overline{U'})$, which is a closed subspace of $C^\infty_c(\Omega)$. By the Hahn-Banach theorem there exists $v \in \mathfrak{D}'(\Omega)$ such that $v|_{\overline{U'}} = u|_{\overline{U'}}$.

Theorem 6. *P is hypoelliptic on Ω if and only if for all $\Omega' \subset \Omega$ with $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ there exists for each $x_0 \in \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of x_0 such that P is $*$-hypeolliptic on Ω_{x_0}.*

Proof. Let P be hypoelliptic on Ω. Let $\Omega' \subset \Omega$ be any open subset. Choose $\Omega_{x_0} \subset \Omega'$ to be any relatively compact open neighborhood of x_0. Assume $u \in \mathfrak{D}'_F(\Omega_{x_0})$ and $Pu \in C^\infty(\Omega_{x_0})$. In order to show $u \in C^\infty(\Omega_{x_0})$,

we must find a neighborhood $\Omega_y \subset \Omega_{x_0}$ of each $y \in \Omega_{x_0}$ such that $u \in C^\infty(\Omega_y)$. Let Ω_y be any relatively compact neighborhood of $y \in \Omega_{x_0}$ such that $\overline{\Omega_y} \subset \Omega_{x_0}$. By Theorem 5 there exists $v \in \mathfrak{D}'(\Omega)$ such that $v|_{\overline{\Omega_y}} = u|_{\overline{\Omega_y}}$. Since $Pu \in C^\infty(\Omega_{x_0})$, $Pu = Pv \in C^\infty(\Omega_y)$. Since P is hypoelliptic on Ω, $u = v|_{\Omega_y} \in C^\infty(\Omega_y)$.

Suppose P is $*$-hypoelliptic on each Ω_{x_0} and let $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$. We must show $u \in C^\infty(\Omega_{x_0})$ for all the given $\Omega_{x_0} \subset \Omega'$. Since Ω_{x_0} is relatively compact, $u \in \mathfrak{D}'_F(\Omega_{x_0})$. Also $Pu \in C^\infty(\Omega_{x_0})$, and since P is $*$-hypoelliptic on Ω_{x_0}, $u \in C^\infty(\Omega_{x_0})$.

Theorem 7. *P is local on Ω if and only if for all $\Omega' \subset \Omega$ with $u \in \mathfrak{D}'(\Omega)$ and $Pu \in C^\infty(\Omega')$ there exists for each $x_0 \subset \Omega'$ a relatively compact open neighborhood $\Omega_{x_0} \subset \Omega'$ of $x_0 \in \Omega'$ such that P is $*$-local on Ω_{x_0}.*

Proof. Let P be local on Ω. Let $\Omega' \subset \Omega$ be any open subset and choose $\Omega_{x_0} \subset \Omega'$ to be any relatively compact open neighborhood of $x_0 \in \Omega'$. Assume $u \in \mathfrak{D}'_F(\Omega_{x_0})$ and $Pu \in C^\infty(\Omega_{x_0})$. Choose (Ω_{y_i}) to be an open locally finite cover of Ω_{x_0} such that $\overline{\Omega_{y_i}} \subset \Omega_{x_0}$. By Theorem 5 there exists $v \in \mathfrak{D}'(\Omega)$ such that $v|_{\overline{\Omega_{y_i}}} = u|_{\overline{\Omega_{y_i}}}$. Since $Pu = Pv \in C^\infty(\Omega_{y_i})$ and P is local on Ω, $P(\psi\phi u) \in C^\infty(\Omega_{y_i})$, where $\psi \in C_c^\infty(\Omega_{y_i})$. Let (ψ_i) be a partition of unity subordinate to (Ω_{y_i}).

$$P(\phi u) = P\left(\sum_{i=1}^{M<\infty} \psi_i \phi u\right) = \sum_{i=1}^{M<\infty} P(\psi_i \phi u) \in C^\infty(\Omega_{x_0}).$$

The proof of the converse is given after the definitions of local and $*$-local operators on Ω.

Theorem 7 gives an interpretation of estimates (C) in Theorem 2. Estimates (B) in Theorem 2 imply the existence of local fundamental solutions for the transpose of P. For the proof of this see Trèves [9]. In fact, it is easy to prove that hypoelliptic operators on Ω are characterized by the existence of local fundamental solutions, which may be extended continuously to $\mathfrak{E}'(\Omega_{x_0})$, of the transpose of P and by P being local on Ω.

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