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SOME PROBLEMS ON B_r-COMPLETENESS

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ABSTRACT. In this paper, we give examples to show the following:
1. The product of two B-complete spaces is not necessarily B_r-complete.
2. The Mackey dual of a strict LF-space is not necessarily B_r-complete.
3. A separable, reflexive, and strict LF-space is not, in general, B_r-complete. The second point has reference to a problem of Dieudonné and Schwartz which asks essentially whether the Mackey dual of a strict LF-space is B-complete and which was answered in the negative by Grothendieck.

1. **Introduction.** Summers [8] and Iyahen [6] have shown that the product of two B-complete spaces is not necessarily B-complete. We give here two examples to show that the product of two B-complete spaces need not be B_r-complete, thus answering a question raised by Van Dulst [2], [3]. One of these is a space introduced by Grothendieck [4] and it gives us the first example of a separable, reflexive and strict LF-space (in fact the product of a separable and reflexive F-space and the direct sum of countable copies of a reflexive and separable Banach space) which is not B_r-complete. This is interesting since the countable direct sum of reflexive Banach spaces is B-complete. In [1], Dieudonné and Schwartz raised the following problem:

Let $E = \lim \operatorname{ind} E_n$ *be a strict* LF-*space. Let* H *be a subspace of* E *such that* $H \cap E_n$ *is closed for all* n. *Is* H *necessarily closed?*

In the light of Corollary 2 below, this is equivalent to asking whether the Mackey dual of a strict LF-space is B-complete. Grothendieck [4] gave a counterexample, but it did not answer the following more inclusive question:

Is the Mackey dual of a strict LF-*space even* B_r-*complete?*

The same example as mentioned above is used here to show that the answer to the above question is negative in general.

The other example (a space introduced by Köthe [7]) further shows that

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a separable, nuclear and complete locally convex space need not be B_τ-complete.

2. **Preliminary results.** In this section, we give some criteria for B- and B_τ-completeness which will be used in the sequel. Notations and terminology may be found in [5].

Theorem 1. *Let (E, u) be a quasi-complete locally convex space such that each absolutely convex compact subset is metrizable. Then (E', τ) (where τ is any topology lying between the tolopogy c of uniform convergence on absolutely convex compact subsets of (E, u) and the Mackey topology $\tau(E', E)$) is B-complete (B_τ-complete) iff every sequentially closed (dense) subspace of (E, u) is closed.*

Proof. It is sufficient to prove the result for (E', c). Now $(E', c)' = E$. The condition of metrizability implies that each sequentially closed subspace of (E, u) is almost closed.

Conversely, let $L \subseteq E$ be almost closed and $\{x_n; n \in N\} = K$ be a sequence in L converging to $x \in E$. Then the closed absolutely convex hull B of K is compact by quasi-completeness and $L \cap B = L \cap (B^\circ)^\circ$ is closed since, by hypothesis, L is almost closed. But this implies that $x \in L$ and hence L is sequentially closed.

Note. In the above, it is clear that quasi-completeness can be replaced by the following condition:

The closed absolutely convex hull of every convergent sequence is compact.

It should be noted that sequentially closed subspaces need not, in general, be almost closed. Let $E = \Pi\{l_2^\alpha; \alpha \in \Lambda\}$ where each l_2^α is a copy of l_2 and cardinality $|\Lambda| \geq c$ (where c is the cardinality of the continuum). It is easily seen that if H is the subspace of E consisting of all elements of E which vanish for all but countable values of α, then H is sequentially closed but not almost closed.

Corollary 1. *Let (E, u) be a quasi-complete locally convex space such that (E', c) is separable. Then (E', τ), $c \leq \tau \leq \tau(E', E)$, is B-complete (B_τ-complete) iff every sequentially closed (dense) subspace of (E, u) is closed.*

Proof. The condition implies that the absolutely convex compact subsets are weakly metrizable and hence metrizable in the original topology u.

Corollary 2. *Let (E, u) be any strict* LF*-space. Then (E', τ) (same notation as above) is B-complete (B_r-complete) iff every sequentially closed (dense) subspace of (E, u) is closed.*

Corollary 3 (Iyahen [6]). *Let $E = A \times B$ where A is the direct sum of countable copies of l_p and B is the product of countable copies of l_q with $q < p$ and $1/p + 1/q = 1$. Then E is not B-complete.*

Proof. Grothendieck [4], in fact, shows that E contains a sequentially closed subspace H which is not closed. Since E is an LF-space, it follows by Corollary 2 that $(E', \tau(E', E))$ is not B-complete. But $(E', \tau(E', E))$ is topologically isomorphic to (E, u).

Note. It is possible to state results similar to Theorem 1 for locally convex spaces satisfying the Krein-Schmulian theorem or for hypercomplete locally convex spaces by substituting the words "convex set" or "absolutely convex set", respectively, in place of the word "subspace."

3. **Counterexamples.** The first counterexample, given below in Theorem 2, uses a space introduced in [4].

Theorem 2. *Let $A = \bigoplus_{n=1}^{\infty} l_p^n$ and $B = \prod_{n=1}^{\infty} l_q^n$, where l_p^n and l_q^n are copies of l_p and l_q respectively with $1 < q < p$ and $1/p + 1/q = 1$. Let $G = A \times B$.*

Then

(a) *G is a strict* LF*-space.*

(b) *A and B satisfy the Krein-Schmulian theorem and, hence, are B-complete.*

(c) *G is not B_r-complete.*

(d) *$(G', \tau(G', G))$ is not B_r-complete.*

(e) *G contains a sequentially closed dense subspace H which, even with the Mackey topology $\tau(H, H')$, is not bornological.*

Proof. (a) is clear. In fact, $G = \lim \operatorname{ind} F_n$, where $F_n = (l_p \times l_p \times \cdots \times l_p) \times B$ (n factors of l_p).

(b) From Corollary 2 and the remarks following, and since A is the Mackey dual of the Fréchet space B, it follows that A satisfies the Krein-Schmulian theorem. B also satisfies it since B is a Fréchet space.

(c) As in Corollary 3, it is sufficient to prove (d).

(d) By Corollary 2, it suffices to construct a sequentially closed dense subspace L in G which is not closed.

Construction. Let $e_k = \{\delta_{kn}\}_{n \in N}$ where

$$\delta_{kn} = 0 \quad \text{if } k \neq n,$$
$$= 1 \quad \text{if } k = n.$$

Then $e_k \in l_p \cap l_q$. Define $e_{ik} = \{\delta_{in} e_k\}_{n \in N}$. Then e_{ik} lies in $A \cap B$. Let $a_{ik} = (e_{ik}, e_{2i-1,k})$. Define

$$b_{jkn} = (x_{jkn}, e_{jk} + ((jkn)!)^{((jkn)!)} e_{2j,\psi(k,n)}),$$

where ψ: $N \times N \to N$ is one-one and onto. We next define x_{jkn}. Let θ: $N \times N \times N \to N$ be one-one and onto. Let $S_{kn} = \theta[(k, n) \times N] \subseteq N$. Then each S_{kn} is a countably infinite set and $S_{kn} \cap S_{rt} = \varnothing$, if $(k, n) \neq (r, t)$. Choose an element $x_{kn} \in l_p - l_q$ (set-theoretic difference) for each $(k, n) \in N \times N$ such that x_{kn} is zero outside S_{kn} and has p-norm not exceeding $1/n$. Such a choice is possible since l_q is a dense proper subspace of l_p. Define

$$x_{jkn} = \{\delta_{jr} x_{kn}\}_{r \in N}.$$

Then $x_{jkn} \in A$ and $\lim_{n\to\infty} |x_{jkn}|_p = 0$ where $|\cdot|_p$ denotes the usual p-norm.

Let L be the set of all elements x of G which are of the form:

(1) $$x = \sum_{i,k=1}^{\infty} \mu_{ik} a_{ik} + \sum_{j,k,n=1}^{\infty} \lambda_{jkn} b_{jkn},$$

where μ_{ik} and λ_{jkn} are scalars and

(2) $$\sum_{k=1}^{\infty} |\mu_{ik}|^q < \infty \quad \text{for each } i,$$

(3) $$\sum_{k,n=1}^{\infty} |\lambda_{jkn}|^q ((jkn)!)^{((jkn)!)\cdot q} < \infty \quad \text{for each } j,$$

and there exists for each such x a fixed j_0 (depending on x) such that $\mu_{ik} = \lambda_{jkn} = 0$ for $i, j > j_0$.

From (3), it follows easily that

(4) $$\sum_n |\lambda_{jkn}| < \infty \quad \text{for each fixed } j \text{ and } k,$$

and

$$\sum_k \left(\sum_n |\lambda_{jkn}|\right)^q < \infty \quad \text{for each fixed } j. \tag{5}$$

Let i_0 and j_0 be the least values of i and j, respectively, in (1) such that $\mu_{ik} = 0$ for $i > i_0$, and $\lambda_{jkn} = 0$ for $j > j_0$. Let $z_0 = \max(i_0, j_0)$.

Following Köthe [7], we may write the elements x of G in the form:

$$x = (\cdots, x_{-n}, \cdots, x_{-2}, x_{-1} || x_1, \cdots, x_n, \cdots) \tag{6}$$

where $x_i \in l_q$ and $x_{-i} \in l_p$ and all but a finite number of $x_{-i} = 0$.

If x in (1) is written in the above form, then it is not difficult to see that $x_{-z_0} \neq 0$.

This is clear, if $i_0 \neq j_0$. In case $i_0 = j_0$, the same result is true since the sum of an element in $l_p - l_q$ and an element in l_q cannot be zero.

Let

$$x_r = \sum \mu^r_{ik} a_{ik} + \sum \lambda^r_{jkn} b_{jkn} \in L \tag{7}$$

converge to an element $x \in G$. If i_r and j_r are defined for x_r (as i_0 and j_0 for x above), then convergence of x_r implies, by the properties of the inductive limit topology of LF-spaces [1, Proposition 4], that there exists a fixed positive integer j_0 such that

$$z_r = \max(i_r, j_r) < j_0 \quad \text{for all } r.$$

Let L_m be the set of all such sums in (1) for which $\mu_{ik} = \lambda_{jkn} = 0$ for $i, j > m$ and any k, n. Then we have $L = \bigcup\{L_m;\ m \in N\}$. In order to show that L is sequentially closed, it suffices to show that L_m is sequentially closed for all $m \geq 1$. This we do by induction. Assume L_{m-1} sequentially closed and let x_r as in (7) be a sequence in L_m converging to $x \in G$. Then we can write x_r in the following form:

$$x_r = \sum_{i \leq m-1, k} \mu^r_{ik} a_{ik} + \sum_{j \leq m-1, k, n} \lambda^r_{jkn} b_{jkn} + x'_r \quad \text{(say)}$$

where

$$x'_r = \sum_k \mu^r_{mk} a_{mk} + \sum_{k,n} \lambda^r_{mkn} b_{mkn}.$$

Let x_r be represented in the form (6) and let x^r_t denote the element in the tth position where t is a positive or negative integer. The convergence of the sequence x_r implies the convergence in l_q of the sequence x^r_{2m}.

Since

$$x^r_{2m} = \sum_{k,n} \lambda^r_{mkn} d_{mkn} e_{2m,\psi(k,n)}$$

(where d_{mkn} equals $(mkn)!$ raised to the power $(mkn)!$) converges in l_q and since convergence in l_q implies coordinate-wise convergence, it follows that the limit element $\bar{x}_{2m}$ is of the form:

$$\bar{x}_{2m} = \sum_{k,n} \lambda_{mkn} d_{mkn} e_{2m,\psi(k,n)}$$

where $\lambda_{mkn} = \lim_r \lambda^r_{mkn}$ for each k, n. This implies that for any $\epsilon > 0$, there exists a positive integer R such that for $r > R$, we have

$$|x^r_{2m} - \bar{x}_{2m}|^q_q = \sum_{k,n} |\lambda^r_{mkn} - \lambda_{mkn}|^q d^q_{mkn} < \epsilon. \tag{8}$$

Similarly, for the sequence x'_r, the element x'^r_m at the mth position is given by:

$$x'^r_m = \sum_{k,n} \lambda^r_{mkn} e_{mk} = \sum_k \left(\sum_n \lambda^r_{mkn} \right) e_{mk}.$$

(Recall by (4) $\Sigma_n |\lambda^r_{mkn}|$ is convergent.) Using inequality (8), it can be shown without much difficulty that the sequence x'^r_m converges to $\bar{x}'_m$ in l_q where $\bar{x}'_m = \Sigma_k (\Sigma_n \lambda_{mkn}) e_{mk}$.

Again, the convergence of the sequence x_r in G implies that the sequence x^r_{2m-1} converges in l_q. Since $x^r_{2m-1} = \Sigma_k \mu^r_{mk} e_{2m-1,k}$, it follows by an argument essentially similar to that for the sequence x^r_{2m}, that the limit element $\bar{x}_{2m-1}$ is of the form $\Sigma_k \mu_{mk} e_{2m-1,k}$, where $\mu_{mk} = \lim_r \mu^r_{mk}$.

Let us next look at the $-m$th position. Using the same notation,

$$x^r_{-m} = \sum_k \mu^r_{mk} e_{mk} + \sum_{k,n} \lambda^r_{mkn} x_{mkn}.$$

The sequence $\Sigma_k \mu^r_{mk} e_{mk}$ certainly converges in l_p since we have seen that it converges in l_q, and the limit element is $\Sigma_k \mu_{mk} e_{mk}$. Also

$$\left| \sum_{k,n} \lambda^r_{mkn} x_{mkn} - \sum_{k,n} \lambda_{mkn} x_{mkn} \right|^p_p = \sum_{k,n} |\lambda^r_{mkn} - \lambda_{mkn}|^p |x_{mkn}|^p_p$$

$$\leq \sum_{k,n} |\lambda^r_{mkn} - \lambda_{mkn}|^p,$$

but the right-hand side tends to zero by the inequality (8).

The above arguments thus imply that the sequence x'_r converges in G

to the element $\Sigma_k \mu_{mk} a_{mk} + \Sigma_{k,n} \lambda_{mkn} b_{mkn} \in L_m$ (since it can be seen without much difficulty that conditions (2) and (3) are satisfied). Hence, by the induction hypothesis, it follows that the sequence $x_r - x_r'$ in L_{m-1} converges to some element in L_{m-1} and hence, the sequence x_r itself converges to some element in L_m.

A similar type of argument shows that L_1 is also sequentially closed. Note that since the relative topology in the finite products of l_p and l_q which contains L_m is metric, it is clear that L_m is closed in G for all m.

That L is proper is clear since in G there exist elements of the form $(\cdots, x_{-n}, \cdots, x_{-2}, x_{-2}, x_{-1} \| x_1, x_2, \cdots, x_n, \cdots)$ where $x_n \neq 0$ for infinitely many values of n.

We next show that L is dense. Let E_j be defined as follows: $E_j = A \times l_q \times \cdots \times l_q$ (E_j has j factors of l_q). Let π_j be the projection map of G onto E_j. Then inductively, it can be seen as in [7] (by using the fact that $\lim_n x_{jkn} = 0$ in l_p for each j, k) that $(\pi_j L)^0 = 0$ for each j. This clearly implies that L is dense.

(e) Let $H = L \oplus [x_1]$ where x_1 is an element of G such that $x_1 \notin L$ and $[x_1]$ is the subspace generated by x_1. It is easy to show that H is also sequentially closed in G. Let $f \neq 0$ be a linear functional on H with the kernel L. Then by Webb [9], f is sequentially continuous. If H with its Mackey topology is bornological, then f is continuous on H and so has a unique continuous extension $\bar{f}$ to G. But then the kernel $N(\bar{f})$ of $\bar{f}$ is a closed subspace of G containing L and as L is dense, it follows that $N(\bar{f}) = G$. This means that $\bar{f} = 0$, which is a contradiction. This completes the proof of Theorem 2.

The other counterexample is given below in Theorem 3.

Theorem 3. *Consider the space $\phi\omega \times \omega\phi$ given in* [7]. *Then we have the following:*

(a) *$\phi\omega$ and $\omega\phi$ are separable, nuclear and B-complete locally convex spaces.*

(b) *$\phi\omega \times \omega\phi$ is a separable, nuclear and complete locally convex space.*

(c) *$\phi\omega \times \omega\phi$ is not B_r-complete.*

Proof. Clearly, $\phi\omega$ and $\omega\phi$ are separable, nuclear and complete locally convex spaces. Now the Mackey dual of $\phi\omega$ is $\omega\phi$ and that of $\omega\phi$ is $\phi\omega$. Köthe [7] shows that any sequentially closed subspace in these spaces is always closed. So, by Corollary 1, $\phi\omega$ and $\omega\phi$ are B-complete.

(b) This is clear from (a).

(c) Köthe [7] has constructed a dense and sequentially closed subspace in the space $\phi\omega \times \omega\phi$ which is not closed. If E denotes the space $\phi\omega \times \omega\phi$, then by (a), $(E', \tau(E', E))$ is topologically isomorphic to E, and since E is separable, it follows by Corollary 1 that $(E', \tau(E', E))$ and hence E is not B_r-complete.

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