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ON DUGUNDJI'S NOTION OF POSITIVE DEFINITENESS

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ABSTRACT. Dugundji's notion of positive definiteness is generalized to nonnegative real-valued functions on a uniform space. Its relations with completeness and various notions of compactness are investigated. For an arbitrary uniform space X, there may be lack of the right kind of lower semicontinuous real-valued functions on X and so a further generalization of Dugundji's notion of positive definiteness is needed for the development of the fixed point (or coincidence) theory. With such an extension, a very general fixed point theorem is obtained to include a recent result of the author, which contains, as special cases, some results of S. Banach, F. E. Browder, D. W. Boyd and J. S. W. Wong, M. Edelstein and R. Kannan.

1. **Introduction.** Let P be a nonnegative real-valued function on a metric space (X, d). Let A be subset of X. For any positive real number r, let

$$P_A(r) = \inf\{P(x): d(x, A) \geq r\}$$

($\inf \varnothing = \infty$). Then P_A is a monotone nondecreasing function of $(0, \infty)$ into $[0, \infty]$. P is *positive definite* mod A if $P_A(r)$ is positive for all r in $(0, \infty)$. This notion of positive definiteness was recently introduced by J. Dugundji [4]. Let $X \times X$ be endowed with the metric D defined by

$$D((x, y), (x', y')) = d(x, x') + d(y, y'), \qquad x, y, x', y' \in X.$$

Let $\Delta(X)$ denote the diagonal of $X \times X$. It was observed in [4] that a function P of $X \times X$ into $[0, \infty)$ is positive definite mod $\Delta(X)$ if and only if for

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any $r > 0$, there exists $\delta(r) > 0$ such that $d(x, y) < r$ whenever $P(x, y) < \delta(r)$. For any function V of a set X into $[0, \infty)$, P_V will denote the function on $X \times X$ defined by

$$P_V(x, y) = V(x) + V(y), \qquad x, y \in X.$$

The following result was obtained in [4].

Theorem 1 (J. Dugundji). *Let (X, d) be a metric space. Then the following conditions are equivalent.*

(a) *X is complete.*

(b) *For every lower semicontinuous function V of X into $[0, \infty)$ such that* $\inf V(X) = 0$ *and P_V is positive definite* mod $\Delta(X)$, $V(x) = 0$ *for some x in X.*

Moreover, if (b) *is satisfied and if $\{x_n\}$ is a sequence in X such that $\{V(x_n)\}$ converges to 0, then $\{x_n\}$ converges to x.*

In order to obtain a fixed point for a function T of X into X, it is only natural to assume that V in Theorem 1 is the function on X such that $V(x) = d(x, T(x))$ for all x in X.

Theorem 2. *Let (X, d) be a complete metric space. Let T be a self map on X. Suppose that*

(a) *the function V defined by $V(x) = d(x, T(x))$, $x \in X$, is lower semicontinuous;*

(b) $\inf V(X) = 0$;

(c) *P_V is positive definite* mod $\Delta(X)$.

Then T has a fixed point.

The above consequence of Theorem 1 was obtained earlier by the author in 1969 [8] and was published in [10]. Instead of using Dugundji's notion of positive definiteness, the author assumes that T is a uniformly continuous function of $(X, {}_Td)$ into (X, d), where

$${}_Td(x, y) = \max\{d(x, T(x)), d(y, T(y))\} \quad \text{if } x, y \in X, \ x \neq y;$$

$${}_Td(x, x) = 0 \quad \text{if } x \in X.$$

However, T is a uniformly continuous function of $(X, {}_Td)$ into (X, d) if and only if P_V is positive definite mod $\Delta(X)$, for, instead of using ${}_Td$, one may, without loss of generality, use ${}^Td$:

$${}^Td(x, y) = d(x, T(x)) + d(y, T(y)) \quad \text{if } x, y \in X, \ x \neq y;$$

$${}^Td(x, x) = 0 \quad \text{if } x \in X.$$

Perhaps we should also mention here that the argument in Dugundji's proof for (a) $\Rightarrow$ (b), Theorem 1 is similar to the argument in the author's proof for Theorem 2 in 1969 [8], [10]. In [4], Dugundji proved that a result of F. E. Browder [2] follows from Theorem 1. It was proved by the author in 1969 that this result of Browder follows from Theorem 2 (and therefore Theorem 1) [8], [10].

In this paper, we shall generalize Dugundji's notion of positive definiteness to nonnegative real-valued functions on a uniform space and investigate its relations with completeness and various notions of compactness.

2. **Positive definite functions.** Let $(X, \mathfrak{U})$ be a uniform space. Let P be a function of X into $[0, \infty)$. Let A be a subset of X. For any U in $\mathfrak{U}$, let

$$P_A(U) = \inf\{P(x): x \in X,\ x \notin U[A]\}.$$

Then P_A is a function of $\mathfrak{U}$ into $[0, \infty]$. P is *positive definite* mod A if $P_A(U) > 0$ for all U in $\mathfrak{U}$. Let P, Q be functions of X into $[0, \infty)$. Then (a) P_A is monotone, i.e. $P_A(U) \leq P_A(V)$ if $U, V \in \mathfrak{U}$ and if $U \subset V$, (b) rP is positive definite mod A if P is positive definite mod A and if $r > 0$, (c) PQ and $P \wedge Q$ are positive definite mod A if both P and Q are positive definite mod A, (d) Q is positive definite mod A if P is positive definite mod A and if $P \leq Q$. Hence $P \vee Q$, $P + Q$ are positive definite mod A if at least one of P, Q is positive definite mod A.

Lemma 1. *Let $(X, \mathfrak{U})$ be a uniform space, let A be a subset of X and let P be a function of X into $[0, \infty)$ which is positive definite* mod A. *Then* $P^{-1}(\{0\}) \subset \mathrm{cl}\, A$.

Proof. Let $x \in P^{-1}(\{0\})$. Since P is positive definite mod A, $x \in U[A]$ for all U in $\mathfrak{U}$. So $x \in \bigcap\{U[A]: U \in \mathfrak{U}\}$, i.e. $x \in \mathrm{cl}\, A$.

Lemma 2. *Let $(X, \mathfrak{U})$ be a uniform space. Let $X \times X$ be endowed with the product uniformity $\mathfrak{U} \times \mathfrak{U}$. Let P be a function of $X \times X$ into $[0, \infty)$. Then the following conditions are equivalent.*

(a) *P is positive definite* mod $\Delta(X)$.

(b) *For any U in $\mathfrak{U}$, there exists $\delta(U) > 0$ such that $P^{-1}((0, \delta(U))) \subset U$.*

Proof. (a) $\Rightarrow$ (b). Suppose to the contrary that there exists U in $\mathfrak{U}$ such that for any $r > 0$, there exists x_r in $X \times X$ for which $x_r \notin U$ and $P(x_r) < r$. Then $x_r \notin U \times U[\Delta(X)]$ for all $r > 0$. So $P_{\Delta(X)}(U \times U) = 0$, a contradiction to (a).

(b) $\Rightarrow$ (a). Let $U \in \mathfrak{U}$. Since $P_{\Delta(X)}$ is monotone and $\{V \times V: V \in \mathfrak{U}\}$ is a filter base for $\mathfrak{U} \times \mathfrak{U}$, it suffices to prove that $P_{\Delta(X)}(U \times U) > 0$. Suppose to the contrary that $P_{\Delta(X)}(U \times U) = 0$. Then there exists a sequence $\{x_n\}$ in $X \times X$ such that $\{P(x_n)\}$ converges to 0 and $x_n \notin U \times U[\Delta(X)]$ for each n. Since $\Delta(X) \subset U$, $x_n \notin U$ for each n. So for any $r > 0$, $\{x_n\}$ is eventually in $P^{-1}((0, r))$ but none of the x_n's belongs to U. Hence $P^{-1}((0, r))$ is not contained in U for all $r > 0$, a contradiction to (b).

Theorem 3. *Let $(X, \mathfrak{U})$ be a complete Hausdorff uniform space. Let V be a lower semicontinuous function of X into $[0, \infty)$ such that* $\inf V(X) = 0$. *Suppose that P_V is positive definite* mod $\Delta(X)$. *Then $V(x) = 0$ for a unique x in X. Moreover, if $\{x_n\}$ is a net in X such that $\{V(x_n)\}$ converges to 0, then $\{x_n\}$ converges to x.*

Proof. Consider $F_n = \{x \in X: V(x) \leq 1/n, n = 1, 2, \cdots\}$. Since V is lower semicontinuous, each F_n is closed. Since $\inf V(X) = 0$, each F_n is nonempty. We shall now prove that the family $\mathfrak{F} = \{F_n: n = 1, 2, \cdots\}$ contains small sets. Let $U \in \mathfrak{U}$. Since P_V is positive definite mod $\Delta(X)$, by Lemma 2, there exists $\delta(U) > 0$ such that $P_V^{-1}((0, \delta(U))) \subset U$. Choose n such that $n > 2/\delta(U)$. Let $x, y \in F_n$. Then $P_V(x, y) < \delta(U)$. So $(x, y) \in U$. Thus $F_n \times F_n \subset U$ and $\mathfrak{F}$ contains small sets. Since $(X, \mathfrak{U})$ is complete and $\mathfrak{F}$ has the finite intersection property, $\mathfrak{F}$ has nonempty intersection. Let $x \in \bigcap \mathfrak{F}$. Then $V(x) = 0$.

Now suppose that $V(x) = V(y) = 0$. Then $P_V(x, y) = 0$. Since P_V is positive definite mod $\Delta(X)$, by Lemma 1, $(x, y) \in \operatorname{cl} \Delta(X)$. Since X is Hausdorff, $\Delta(X)$ is closed. So $(x, y) \in \Delta(X)$, i.e. $x = y$.

Now assume that $\{x_n\}$ is a net in X such that $\{V(x_n)\}$ converges to 0. Then the net $\{P_V(x_n, x_m)\}$ converges to 0. Since P_V is positive definite mod $\Delta(X)$, $\{(x_n, x_m)\}$ is eventually in U for all U in $\mathfrak{U}$, i.e. $\{x_n\}$ is a Cauchy net. By completeness of $(X, \mathfrak{U})$, $\{x_n\}$ converges to some z in X. Now $z \in \operatorname{cl} F_n = F_n$ for each n. So $z \in \bigcap \mathfrak{F}$, i.e. $V(z) = 0$. By the uniqueness of the element in $V^{-1}(\{0\})$, $z = x$.

3. Positive definiteness and compactness.

Theorem 4. *Let $(X, \mathfrak{U})$ be a uniform space, let A be a compact subset of X. Let V be a lower semicontinuous function of X into $[0, \infty)$ such that* $\inf V(X) = 0$. *Suppose that V is positive definite* mod A. *Then $V(x) = 0$ for some x in A.*

The following result is a partial converse to the above theorem.

Theorem 5. *Let $(X, \mathfrak{U})$ be a uniform space and let A be a closed subset of X. Suppose that X is normal and for every lower semicontinuous function V of X into $[0, \infty)$ such that V is positive definite* mod A *and* $\inf V(X) = 0$, $V(x) = 0$ *for some x in A. Then A has the Bolzano-Weierstrass property.*

Proof. Suppose to the contrary that A has no Bolzano-Weierstrass property. Then there exists a countable infinite subset B of A which has no limit point. Let f be a bijection of the set Z^+ of all positive integers onto B. Then the function g on B defined by $g(f(n)) = 1/n$, $n = 1, 2, \cdots$, is a continuous function of B into $(0, 2)$. Since B is closed in A and A is closed in X, B is closed in X. Since X is normal, by Tietze's extension theorem, g can be extended to a continuous function h of X into $(0, 2)$ [3, p. 151]. We may assume that $\mathfrak{U}$ is induced by a family $\{d_i : i \in I\}$ of pseudo metrics d_i for X such that $d_i \leq 1$ for each i in I (replace each d_i by $d_i/(1 + d_i)$ if necessary). Let V be the function on X such that $V(x) = h(x) + \sup\{d_i(x, A): i \in I\}$ for all x on X. Since each $d_i(\ , A)$ is continuous on X and $V < 3$, V is a lower semicontinuous function of X into $[0, 3)$. Since V_A is monotone, it is straightforward to prove that V is positive definite mod A. Also, $\inf V(X) \leq \inf V(B) = 0$, i.e. $V(X) = 0$. However, $V(x) \geq h(x) > 0$ for all x in X, a contradiction to the hypothesis.

We owe Dugundji for his versions of the above results for metric spaces. We would like to emphasize here that unlike the case when $\mathfrak{U}$ is induced by a metric for X, in Theorem 5, the phrase "A has the Bolzano-Weierstrass property" cannot be replaced by "A is compact"; so the converse of Theorem 4 is not true. To see this, let us consider the following special case of Theorem 4, where $A = X$. In this case, every nonnegative real-valued lower semicontinuous function on X is positive definite mod A; so Theorem 4 is essentially the following well-known result: Every lower semicontinuous real-valued function on a compact topological space has a minimum value. We shall now give a simple counterexample to the converse of this result (and therefore Theorem 4). Let X be the subspace $[0, \Omega)$ of the ordinal space $[0, \Omega]$, where Ω is the first ordinal of uncountability. Then X is not compact but every real-valued lower semicontinuous function on X has a minimum value. Since X is countably compact, sequentially compact and has the Bolzano-Weierstrass property, it is only natural to find out which of these three notions of compactness can be characterized by real-valued lower semicontinuous functions. This motivates the following characterization of countable compactness.

Theorem 6. *Let X be a topological space. Then the following conditions are equivalent.*

(a) *X is countably compact.*

(b) *Every real-valued lower semicontinuous function on X has a minimum value.*

(c) *Every real-valued lower semicontinuous function on X which is bounded below has a minimum value.*

(d) *Every real-valued lower semicontinuous function on X is bounded below.*

The proof of the above theorem will appear in [12]. A characterization of G_δ sets by means of lower semicontinuous real-valued functions can also be found in [12]. It may be worthwhile to note here that similar results can be obtained by considering upper semicontinuous real-valued functions.

4. **Further extension of Dugundji's notion of positive definiteness.** In view of Theorem 6, there may be a lack of the right kind of real-valued lower semicontinuous functions on an arbitrary uniform space. This and Theorem 3 suggest the need to replace the notions of semicontinuity and positiveness by some more general notions. Let $(X, \mathfrak{U})$ be a uniform space. Let V be a function of X into $X \times X$. $\mathfrak{U}_V$ will denote the uniformity for X with $\{(V^{-1}(U) \times V^{-1}(U)) \cup \Delta(X): U \in \mathfrak{U}\}$ as its filter base. Our further results are justified by the following proposition whose proof is straightforward.

Proposition. *Let (X, d) be a metric space. Let X be endowed with the uniformity $\mathfrak{U}$ induced by d. Let $X \times X$ be endowed with the product uniformity $\mathfrak{U} \times \mathfrak{U}$. Let T be a self map on X. Let V be the function on X defined by $V(x) = (x, T(x))$, $x \in X$. Let P be the map on $X \times X$ such that*

$$P(x, y) = d(x, T(x)) + d(y, T(y)), \qquad x, y \in X.$$

Then the following conditions are equivalent.

(a) *P is positive definite* mod $\Delta(X)$.

(b) *T is a uniformly continuous function of $(X, {}_T d)$ into (X, d).*

(c) $\mathfrak{U} \subset \mathfrak{U}_V$.

Theorem 7. *Let $(X, \mathfrak{U})$ be a complete Hausdorff uniform space. Let V be a function of X into $X \times X$ such that $\mathfrak{U} \subset \mathfrak{U}_V$. Suppose that there exists a base $\mathfrak{B}$ of closed sets for $\mathfrak{U}$ such that $V^{-1}(U)$ is a nonempty closed subset of X for each U in $\mathfrak{B}$. Then there exists a unique x in X such that $V(x) \in \Delta(X)$. Moreover, if $\{x_n\}$ is a net in X such that $\{V(x_n)\}$ is eventually in every member of $\mathfrak{U}$, then $\{x_n\}$ converges to x.*

Proof. Let $\mathfrak{F} = \{V^{-1}(U): U \in \mathfrak{B}\}$. Then $\mathfrak{F}$ is a family of nonempty closed

subsets of X which has the finite intersection property. Since $\mathfrak{U} \subset \mathfrak{U}_V$, $\mathfrak{F}$ contains small sets. By completeness of $(X, \mathfrak{U})$, $\mathfrak{F}$ has nonempty intersection. Let $x \in \bigcap \mathfrak{F}$. Then $V(x) \in U$ for all U in $\mathfrak{B}$. Since $(X, \mathfrak{U})$ is Hausdorff, $\bigcap \mathfrak{B} = \Delta(X)$. So $V(x) \in \Delta(X)$.

Now let $\{x_n\}$ be a net in X such that $\{V(x_n)\}$ is eventually in every member of $\mathfrak{U}$. Let $W \in \mathfrak{U}$. Since $\mathfrak{U} \subset \mathfrak{U}_V$, there exists U in $\mathfrak{U}$ such that $V^{-1}(U) \times V^{-1}(U) \subset W$. Since $\{V(x_n)\}$ is eventually in U, $\{x_n\}$ is eventually in $V^{-1}(U)$. Since $V(x) \in \Delta(X)$, $x \in V^{-1}(\Delta(X)) \subset V^{-1}(U)$. So $\{(x_n, x)\}$ is eventually in $V^{-1}(U) \times V^{-1}(U)$. Therefore $\{(x_n, x)\}$ is eventually in W, i.e. $\{x_n\}$ is eventually in $W[x]$. By varying W in $\mathfrak{U}$, we conclude that $\{x_n\}$ converges to x.

Now suppose that $V(y) \in \Delta(X)$. Let $\{y_n\}$ be the sequence with $y_n = y$ for each n. Then $\{V(y_n)\}$ is eventually in $\Delta(X)$ and therefore eventually in every member of $\mathfrak{U}$. So $\{y_n\}$ converges to x, i.e. $y = x$. Hence there exists a unique x in X such that $V(x) \in \Delta(X)$.

Theorem 8. *Let $(X, \mathfrak{U})$ be a complete Hausdorff uniform space. Let T be a self map on X. Let V be a self map on $X \times X$. Let V_T be the function on X defined by*

$$V_T(x) = V(x, T(x)), \quad x \in X.$$

Suppose that

(a) *there exists a base $\mathfrak{B}$ of closed subsets of $X \times X$ for $\mathfrak{U}$ such that $V_T^{-1}(U)$ is a nonempty closed subset of X for each U in $\mathfrak{B}$;*

(b) $\mathfrak{U} \subset \mathfrak{U}_{V_T}$.

Then there exists a unique x in X such that $V(x, T(x)) \in \Delta(X)$. Hence if further $V^{-1}(\Delta(X)) \subset \Delta(X))$ then T has a unique fixed point.

The above result follows from Theorem 7. It generalizes our theorem in [11] (let V be the identity map on $X \times X$) and therefore generalizes the Banach contraction mapping theorem, a result of M. Edelstein [5, 3.1], a result of F. Browder [2, Corollary to Theorem 1], a result of D. W. Boyd and J. S. W. Wong [1, Theorem 2] and a recent result of R. Kannan [6, Theorem 2].

Added in proof. After [12] was submitted for publication, we learned that the result we cited in Theorem 6 appeared in the Appendix of Jörg Blatter, *Grothendieck spaces in approximation theory*, Mem. Amer. Math. Soc. No. 120 (1972).

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