ABSTRACT. A formula, analogous to the classical Burnside lemma, is developed which counts orbit representatives from a set under a group action with a given stabilizer subgroup conjugate class. This formula is applied in a manner analogous to a proof of Polya's theorem to obtain an enumeration of patterns with a given automorphism group.

1. Let $S$ be a finite set and $G$ a finite group acting on $S$. Let $\Delta$ be a system of orbit representatives for $G$ acting on $S$. The following theorem is well known:

Theorem 1 (Burnside [1]). For any function $\omega$ defined on $S$ satisfying $\omega(\sigma s) = \omega(s)$ for all $\sigma \in G$, for all $s \in S$, we have

$$\sum_{s \in \Delta} \omega(s) = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{s \in S} \omega(s) \chi(\sigma s = s)$$

where

$$\chi(\text{statement}) = \begin{cases} 1 & \text{if statement is true;} \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in S$ let $G_s = \{ \sigma \in G : \sigma s = s \}$ be the stabilizer subgroup of $G$ at $s$. Let $G_1, G_2, \ldots, G_N$ be a complete set of nonconjugate subgroups of $G$, ordered such that $|G_1| \geq \cdots \geq |G_N|$. For any two subgroups $H, K \subseteq G$ we define

$$M_K(H) = \frac{1}{|K|} \sum_{\sigma \in G} \chi(\sigma H \sigma^{-1} \subseteq K).$$

Received by the editors July 1, 1973.


Key words and phrases. Orbit, stabilizer subgroup, conjugate subgroup, mark, pattern inventory, Möbius function.


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$M_K(H)$ is sometimes called the *mark* of $K$ at $H$. The matrix $M = (M_{G_j}(G_i))$ is triangular and $M_{G_j}(G_i) \geq 1$ so that we can define $B = M^{-1}$, $B = (b_{ij})$.

We also note that $M_{K(K)}$ is constant on conjugate subgroups of $G$.

In this paper we show the following result:

**Theorem 2.** For any function $\omega$ defined on $S$ satisfying $\omega(\sigma s) = \omega(s)$ for all $\sigma \in G$, for all $s \in S$, we have

$$\sum_{s \in \Delta} \omega(s) \chi(G_s \sim G_i) = \sum_{j=1}^N b_{ij} \sum_{s \in S} \omega(s) \chi(G_js = s),$$

where $G_s \sim G_i$ means $G_s$ conjugate to $G_i$ and $G_js = s$ means $s$ is fixed by all of $G_j$.

In an elegant paper [2], DeBruijn showed that Pólya’s counting theorem [5] can be obtained from Theorem 1 upon letting $S = R^D$, where $R^D$ is the set of functions from the finite set $D = \{1, 2, \ldots, |D|\}$ to the finite set $R = \{1, 2, \ldots, |R|\}$, letting $G$ act on $D$ and hence on $R^D$ by setting $\sigma f(d) = f(\sigma^{-1}d)$, and setting $\omega(f) = \prod_{d \in D} x_{f(d)}$, where $x_1, x_2, \ldots$ are indeterminates. If we use the same approach, starting from Theorem 2 instead of Theorem 1, with no additional difficulty we obtain a more refined version of Pólya’s theorem.

Let $Q_i(x_1, x_2, \ldots)$ denote the *pattern inventory* for patterns whose automorphism group is conjugate to $G_i$:

$$Q_i(x_1, x_2, \ldots) = \sum_{f \in \Delta} \omega(f) \chi(G_f \sim G_i).$$

Let $P_i(y_1, y_2, \ldots)$ denote the *orbit index monomial*:

$$P_i(y_1, y_2, \ldots) = \prod_{d \in D} y_d^{q_{G_i}(d)},$$

where $q_{G_i}(d)$ is the number of orbits of $G_i$ acting on $D$ of length $d$, and $y_1, y_2, \ldots$ are indeterminates. Then we have

**Theorem 3.**

$$Q_i(x_1, x_2, \ldots) = \sum_{j=1}^N b_{ij} P_j(y_1, y_2, \ldots)$$

where we substitute $\sum_{r \in R} x_r^r$ for $y_i$.

This result was proved independently by Stockmeyer [8]. However, he obtained it only as a by-product of elaborate Möbius function techniques.
We show here that Theorem 3 can be derived by simple algebraic manipulations.

We were led to this result by considering the general isomorph rejection problem in a multilinear setting [9], [10]. In this setting, besides Theorem 3, we have also derived from Theorem 2 a whole variety of results counting patterns with a given automorphism group. In particular, for example, we may let $G$ act on $R$ and $D$ or let $G$ act on $D$ and $H$ act on $R$. Or we may extend $S$ to be a cartesian product of finite function spaces, $G$ acting on each of them. Or we may observe that a theorem of Foulkes [3] is nothing more than Theorem 2 applied to a special function space.

2. We shall first prove Theorem 2. The weight function $\omega$ in this theorem is commonly thought of as a function from $S$ into an algebra, usually the algebra of polynomials.

Proof of Theorem 2. Note that for any subgroup $H \subset G$, $\sum_{s \in G} \chi(\sigma H \sigma^{-1} \subset G_s)$ is constant on orbits of $S$, so if we denote the orbit of $s$ by $O_s$ and recall that $|G| = |G_s| |O_s|$ we have

$$\sum_{i=1}^{N} M_{G_i} (H) \sum_{s \in \Delta} \omega(s) \chi(G_s \sim G_i) = \sum_{s \in \Delta} \frac{\omega(s)}{|G_s| |O_s|} \sum_{r \in G} \chi(r H r^{-1} \subset G_s)$$

$$= \sum_{s \in S} \frac{\omega(s)}{|G_s| |O_s|} \sum_{r \in G} \chi(r H r^{-1} \subset G_s)$$

$$= \frac{1}{|G|} \sum_{r \in G} \sum_{s \in S} \omega(rs) \chi(H \subset G_{rs}) = \sum_{s \in S} \omega(s) \chi(H \subset G_s).$$

Inverting $M$ gives our result.

We shall now use Theorem 2 to prove Theorem 3. The similarities between this proof and the proof of Pólya's theorem in [2] are obvious.

Proof of Theorem 3. Note that

$$Q_i(x_1, x_2, \cdots) = \sum_{j=1}^{N} b_{ij} \sum_{f \in R^D} \omega(f) \chi(G_j f = f).$$

But $G_j f = f$ means $\sigma f = f$ for all $\sigma \in G_j$, or $f(d) = f(\sigma^{-1} d)$ for all $d \in D$, for all $\sigma \in G_j$. Thus, $f$ must be restricted to be constant on the orbits of $G_j$ acting on $D$. We can then define $f$ such that $G_j f = f$ by defining it on each orbit. Thus,

$$\sum_{f \in R^D} \omega(f) \chi(G_j f = f) = \sum_{f \in R^{\text{orb}(G_j; D)}} \prod_{A \in \text{orb}(G_j; D)} x_f^{1[A]}.$$
where $\text{Orb}(G_j : D)$ is the set of orbits of $G_j$ acting on $D$. Using the familiar sum-product interchange gives

$$
\sum_{f \in R^D} \omega(f) \chi(G_j; f = f) = \prod_{A \in \text{Orb}(G_j : D)} \sum_{\rho \in R} x^{|A|}
$$

$$
= \prod_{d \in D} \left( \sum_{\rho \in R} x^d \right)^q = P_j \left( \sum_{\rho \in R} x^r, \sum_{\rho \in R} x^{2r}, \ldots \right). \quad \text{Q.E.D.}
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REFERENCES


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