THE DUAL OF A THEOREM OF BISHOP AND PHELPS

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ABSTRACT. We dualize a theorem of Bishop and Phelps by showing that in the dual of a Banach space the intersection of a weak* closed finite codimensional linear variety and a weak* closed convex subset C contains a norm dense set of weak* support points of C. We use this theorem to obtain a result which is related to an abstract approximation problem of Deutsch and Morris.

If C is a weak* closed convex subset of $E^*$, the dual of a Banach space E, then by a weak* support point of C we mean a point $z^* \in C$ for which there exists $z \in E \setminus \{0\}$ such that $S_C(z) = \langle z, z^* \rangle$. ($S_C$ is the support function for C and is defined for each $x \in E$ by $S_C(x) = \sup \{\langle x, x^* \rangle | x^* \in C\}$.) In [5], Phelps showed that the set of weak* support points of C is large in the sense of the following theorem.

**Theorem 1 [Phelps].** If E is a Banach space and C is a weak* closed convex subset of $E^*$, then the weak* support points of C are norm dense in the norm boundary of C.

As a consequence of Theorem 1 and the following lemma of Bishop-Phelps [1, Lemma 4], we will obtain a dual to [1, Theorem A]. We remark that although [1, Lemma 4] is stated only for Banach spaces, its proof is valid in any topological vector space.

**Lemma 2 [Bishop-Phelps].** Suppose M is a closed subspace of finite codimension in a topological vector space X, and that C is a convex subset of X. Suppose $x_0$ is a support point of $C \cap M$ in the subspace M. Then $x_0$ is a support point of C.

By the polar $C^o$ of a set $C \subseteq E$, we mean the set $\{x^* \in E^* | S_C(x^*) \leq 1\}$. If N is a subspace of E, then $N^\perp$ denotes the annihilator of N in $E^*$, i.e. $N^\perp = \{n^* \in E^* | \langle n, n^* \rangle = 0 \text{ for each } n \in N\}$.

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Theorem 3. Let $C$ be a closed convex subset of the Banach space $E$, and let $N$ be a finite-dimensional subspace of $E$. Suppose $z^* \in N^\perp \cap \text{bdry } C^\circ$. Then for each $\epsilon > 0$ there exists a weak* support point $w^*$ of $C^\circ$ such that $w^* \in N^\perp$ and $\|w^* - z^*\| \leq \epsilon$.

Proof. We identify the Banach spaces $E/N$ and $N^\perp$. Recall that $N^\perp$ with the relative $\sigma(E^*, E)$ topology is topologically isomorphic with $(E/N)^*$ with the $\sigma((E/N)^*, E/N)$ topology. Thus the set $C^\circ \cap N^\perp$ is a $\sigma(N^\perp, E/N)$ closed convex subset of $N^\perp$. If $z^* \in \text{norm bdry } (C^\circ \cap N^\perp)$ in $N$, then by Theorem 1 applied to $N$, there exists a $\sigma(N^\perp, E/N)$ support point $w^*$ of $C^\circ \cap N^\perp$ in $N^\perp$ such that $\|w^* - z^*\| \leq \epsilon$. By Lemma 2 the element $w^*$ is a $\sigma(E^*, E)$ support point of $C^\circ$ in $E^*$.

If $z^* \notin \text{norm bdry } (C^\circ \cap N^\perp)$ in $N^\perp$, then we show that $z^*$ is itself a weak* support point of $C^\circ$. Since $z^* \in \text{bdry } C^\circ$, there exists an element $y^* \in E^\setminus C^\circ$ such that the segment $[z^*, y^*] \subseteq E^*/C^\circ$. (Otherwise, $z^*$ is an element of the core of $C^\circ$, and since $C^\circ$ is closed and $E^*$ is of the second category in itself, the core of $C^\circ$ is equal to the interior of $C^\circ$.) Let $M = \text{span}(N^\perp \cup \{y^*\})$ and note that $N^\perp$ is a hyperplane in $M$; if we show that $N^\perp$ supports $C \cap M$ at $z^*$, then from Lemma 2, we can conclude that the point $z^*$ is a weak* support point of $C^\circ$. It suffices to show that $C^\circ$ is disjoint from the open half space $\{n^* + ry^* | n^* \in N^\perp$ and $r > 0\}$ in $M$ defined by $N^\perp$.

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only remains to show that \( z^* \in P(C) \) and \( S_C(z^*) = 1 \). Since \( z^* \) is clearly a weak* support point of \( C^0 \), there exists \( z \in E \setminus \{0\} \) satisfying \( S_C(z) = \langle z, z^* \rangle \). Because \( z \neq 0 \) there exists \( n^* \in E^* \) such that \( \langle z, n^* \rangle > 0 \). Since \( C \) is bounded, we know \( C^0 \) is radial at \( 0 \); hence there exists \( \lambda > 0 \) such that \( \lambda n^* \in C^0 \) so

\[
0 < \langle z, \lambda n^* \rangle \leq S_C(z).
\]

Without loss of generality we can suppose

\[
S_C(z) = \langle z, z^* \rangle = 1.
\]

Thus \( z \in C \) (by the bipolar theorem) and we have

\[
S_C(z^*) = \langle z, z^* \rangle = 1 \quad \text{since} \quad z^* \in C^0.
\]

This completes the proof.

The preceding proposition is related to an abstract approximation problem of Deutsch and Morris [2] called "property (SAIN)" for "simultaneous approximation and interpolation which is norm preserving." In the present context this property (which we call "property (S)") is the following:

If \( E \) is a Banach space, \( M \) is a dense subset of \( E^* \), and \( N \) is a finite-dimensional subspace of \( E^{**} \), then the triple \((E^*, M, N)\) has property (S) if for each \( \varepsilon > 0 \) and \( x^* \in E^* \), there exists \( z^* \in M \) satisfying

\[
||z^* - x^*|| < \varepsilon, \quad ||z^*|| = ||x^*||, \quad \text{and} \quad z^* - x^* \in N^1.
\]

Deutsch and Morris established in [2, Theorem 2.3] that in case \( M \) is a linear subspace of \( E^* \), then \((E^*, M, N)\) has property (S) only if each element of \( N \) either attains its norm at points of \( M \) or not at all. This raises the question of what happens if \( M \) is the norm dense subset \( P(B) \) of \( E^* \) (\( B \) is the unit ball of \( E \))? Since \( P(B) \) is not in general convex (a standing hypothesis on the set \( M \) in previous theorems about property (S)), the techniques of [2] do not apply. However, as a corollary to Proposition 4, we obtain the following answer to the question raised above.

**Corollary 5.** Let \( E \) be a Banach space and \( B \) the unit ball of \( E \). The triple \((E^*, P(B), N)\) has property (S) for each finite-dimensional subspace \( N \subseteq E \).

We remark that in [4] Lambert showed that in the case where \( E = c_0 \) and \( B \) is the unit ball of \( c_0 \), that the triple \((l_1, P(B), N)\) has property (S) for each finite-dimensional subspace \( N \) of \( l_{\infty} \). That this result does not hold for general Banach spaces \( E \) and finite-dimensional subspaces \( N \) of \( E^* \) is shown by the following example.
Example. There exists a Banach space $E$ and a one-dimensional subspace $N$ of $E^{**}$ such that the triple $(E^*, P(B), N)$ does not have property (S).

Let $E = c_0$ with an equivalent norm such that $E^* = l_1$ is strictly convex; let $x^* \in S(l_1) \setminus P(c_0)$, and choose $x^{**} \in S(E^{**})$ so that $\langle x^{**}, x^* \rangle = 1$, and let $N = Rx^{**}$. Then

$$(x^{**} + (x^{**})^{-1}(0)) \cap B^* = \{x^*\}$$

since $S(l_1)$ is strictly convex; thus

$$S(E^*) \cap P(B) \cap (x^* + (x^{**})^{-1}(0)) = \emptyset.$$ 

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